**Properties of Quasinormal Groups (PQG)**

*Research Article*

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**Abstract:**

The A subgroup of a group is termed permutable (or quasinormal) in if it satisfies the following equivalent conditions:

1. For any subgroup of , (the product of subgroups and ) is a group
2. For any subgroup of , , i.e., and are permuting subgroups.
3. For every , permutes with the cyclic subgroup generated by . In symbols, for every and , there exists and an integer such that .

We say that G=AB is the mutually permutable product of the subgroups A and B if A permutes with every subgroup of B and B permutes with every subgroup of A. We say that the product is totally permutable if every subgroup of A permutes with every subgroup of B.

In this paper we prove the following theorem

Let G=AB be the mutually permutable product of the super soluble subgroups A and B. If CoreG(A∩B)=1, then G is super soluble.

**Keywords:** Quasinormal.

**INTRODUCTION**

All groups considered in this paper are finite. It is known that a group which is the product of two super soluble groups is not necessarily super soluble, even if the two factors are normal subgroups of the group. Baer proved in [3] that if a group G is the product of two normal supersoluble groups and G′ is nilpotent, then G is super soluble. The search for generalisations of Baer’s result has been a fruitful topic of investigation recently (see[5,7]).

Most of the generalisations centre around replacing normality of the factors by different permutability conditions. In [2], Asaad and Shaalan considered products satisfying one of the following conditions. We will follow Carocca [6], and say that G=AB is the mutually permutable product of the subgroups A and B if A permutes with every subgroup of B and B permutes with every subgroup of A. We say that the product is totally permutable if every subgroup of A permutes with every subgroup of B. Essentially, the results by Asaad and Shaalan are devoted to obtaining sufficient conditions for a mutually permutable product of two supersoluble subgroups to be supersoluble. They prove in [2, Theorem 3.8] the following generalisation of Baer’s theorem:

Let G be the mutually permutable product of the supersoluble subgroups A and B. If G′ is nilpotent, then G is supersoluble.They also show that the result remains true if “G′ nilpotent” is replaced by “Bnilpotent”[2, Theorem 3.2]. In addition, they prove [2, Theorem 3.1]: If G is the totally permutable product of the supersoluble subgroups A and B, then G is supersoluble. It is well known that if G=AB is a mutually permutable product of two supersoluble subgroups A and B such that A∩B=1, then the product is in fact totally permutable [6,Proposition 3.5], and therefore G is supersoluble. Our main Theorem is a generalisation of this last property.

**Theorem 1.**

Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. If CoreG(A∩B)=1, then G is supersoluble.

The second aim of the present paper has been to obtain more complete information about the structure of mutually permutable products of two supersoluble groups. As a straight forward consequence of Theorem 1, we have that, in the notation used above, G/CoreG(A∩B) is always supersoluble. Therefore every mutually permutable product of two supersoluble subgroups is metasupersoluble. It is possible to obtain more precise information about its structure, as our second main theorem shows.

**Theorem 2.** Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. Then G/F (G)is supersoluble and metabelian.This last theorem can not be improved easily, as the following example shows.

**Example.** Let S3 be the symmetric group of degree 3, given by

S3=〈α, β:α2=β3=1;βα=β2〉 and let T7 be the non-abelian group of order 73 and exponent 7. Write T7=〈a,b〉with a7=b7=[a,b]7=1 and let c=[a,b]. We have that S3 acts on T7 in the following way: aα=b, bα=a, cα=c−1, aβ=a2, bβ=b4, cβ=c. Thus we can consider the semidirect product G=[T7] S3. Take now the subgroups

 A= T7〈β〉and B=T7〈α〉of G. Clearly both A and B are supersoluble, and it is easy to check that G=AB is the mutually permutable product of A and B. Finally, we show that Theorem 1 provides elementary proofs for the results of Asaad and Shaalan about mutually permutable products.2.

Main results The following four lemmas are needed to prove Theorem 1.

 **Lemma 1**[4, Theorem 2]. If G=AB is the mutually permutable product of the supersoluble subgroups A and B, then G is soluble.

**Lemma 2**. Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. Then, either G is supersoluble or NA < G and NB < G for every minimal normal subgroup N of G.

**Proof**. Assume that G is not supersoluble. Then both A and B are proper subgroups of G. Let N be a minimal normal subgroup of G and for contradiction assume that NA=G. Then, as N is abelian, N∩A is a normal subgroup of

〈N,A〉= G. Since N is a minimal normal subgroup of G and A<G, we have that N∩A=1 and consequently A is a maximal subgroup of G. Clearly we can also assume that B is not contained in A. It is not difficult to argue that we can choose an element b of B\A such that bq∈A for some prime q. Since the product G=AB is mutually permutable, A〈b〉is a subgroup of G and the maximality of A implies that G=A〈b〉. We now take orders to reach our final contradiction:

|A||N|=|G|=|A||〈b〉||A∩〈b〉|=q|A|. Consequently we have that |N|=q and then G is supersoluble, a contradiction.

**Lemma 3**. Let G=AB be the mutually permutable product of the subgroups A and B and let N be any minimal normal subgroup of G. Then either

N∩A=N∩B=1 or N=(N∩A)(N∩B).

**Proof.**  Let N be a minimal normal subgroup of G. Clearly A(N∩B) and (N∩A)B are both subgroups of G. Note that A normalizes N∩(A(N∩B))=(N∩A)(N∩B) and B normalizes N∩((A∩N)B)= (N∩A)(N∩B). Therefore (N∩A)(N∩B) is a normal subgroup of G and the minimality of N yields the result.

**Lemma 4**. Let G be a group, and N a minimal normal subgroup of G such that |N|=pn, where p is a prime and n>1. Denote C=CG(N) and assume that G/Cis supersoluble. Then, if Q/Cis a subgroup of G/C containing Op′(G/C), we have that Q is normal in G and N=∏ti=1Ni, where Ni are non-cyclic minimal normal subgroups of NQ for i=1,...,t.

**Proof.** Since by [8, Lemma A.13.6], we have that Op(G/C)=1 and the commutator subgroup (G/C)′ of G/C is nilpotent because e G/C is supersoluble, it follows that (G/C)′is a p′-group. Therefore (G/C)′is contained in Op′(G/C) and thus Op′(G/C)is a Hall p′-subgroup of G/C. Consequently, Q/Cis a normal subgroup of G/C and hence Q is normal in G. Consider now N as a G-module over GF (p)by conjugation. Then, by Clifford’s Theorem [8, Theorem B.7.3], N viewed as a Q-module is a direct sum N=∏ti=1Ni, where Ni are irreducible Q-modules for i=1,...,t. Suppose that there exists i∈{1,...,t}such that |Ni|=p. Then clearly |Nj|= p for all j. Therefore Q/CQ(Ni) is abelian of exponent dividing p−1, and the same is true for Q/C. In particular, Q/C=Op′(G/C) is a Hall p′-subgroup of G/C. Since N is not cyclic, it follows that Q = G and thus p divides |G/C|. Hence p is the largest prime dividing |G/C|. From the supersolubility of G/C, we obtain that 1= Op(G/C) is a Sylow subgroup of G/C, a contradiction. Consequently, Ni is a non-cyclic minimal normal subgroup of NQ for all i∈{1,...,t},as we wanted to prove.

**Proof of Theorem 1**. Let G=AB be the mutually permutable product of the supersoluble subgroups A and B, with CoreG(A∩B)=1, and suppose that G has been chosen minimal such that its supersoluble residual GU is non-trivial. Let N be a minimal normal subgroup of G contained in GU. Note that N is an elementary abelian p-group for some prime p. Applying Lemma 2, we have that both NA and NB are proper subgroups of G. Moreover, using Lemma 3, we have that either N=(N∩A)(N∩B ) or N∩A=N∩B=1. Assume first that N=(N∩A)(N∩B).

**(i)** If N∩A=1, then N is cyclic. Assume that N∩A=1. It follows that N is contained in B. Let N0 be a non-trivial cyclic subgroup of N. Since AN0 is a subgroup of G, we have that N0 =AN0∩N is anormal subgroup of AN0. Hence every cyclic subgroup of N is normalised by A. Now let N1 be a minimal normal subgroup of B contained in N. Since B is supersoluble, it follows

That N1 is cyclic and thus normalised by A. Hence N1 is a normal subgroup of G. The minimality of N implies that N=N1 and consequently N is cyclic.

**(ii)** N∩A=1 and N∩B=1.On the contrary, assume that N∩A=1. From (i), we know that N is cyclic. Moreover, Nis contained in B. Hence AN∩B= (A∩B)N. Let L=CoreG(A∩B)N). Clearly, N is contained in Land L=L∩((A∩B)N)=(L∩A∩B)N. It is clear that G/L=(AL/L)(BL/L)is a mutually permutable product of AL/L and BL/L suchthat CoreG/L((AL/L)∩(BL/L))=1. By the minimality of G, it follows that G/L is supersoluble. On the other hand, since N is cyclic, we have that G/CG(N) is abelian. Hence G/CL(N) is supersoluble and GUCL(N)=C. Note that C=N×(C∩A∩B). Therefore C∩A∩B contains a Hall p′-subgroup of C. Since CoreG(A∩B)=1 and Op′(C) is a normal subgroup of G contained in C∩A∩B, we have that Op′(C)=1. Moreover, C′=(C∩A∩B)′ is a normal subgroup of G contained in A∩B. Consequently, C′=1 and C is an abelian p-group. In particular, GU is abelian and thus GU is complemented in G by a supersoluble normalizer D which is also a supersoluble projector of G, by [8, Theorems V.4.2 and V.5.18]. Since N is cyclic, we know that N is central with respect to the saturated formation of all supersoluble groups. By [8,Theorem V.3.2.e], Dcovers N and thus N is contained in D. It follows ND∩GU=1, a contradiction.

**(iii)**  Either N=N∩A or N=N∩B. If we have N=N∩A=N∩B, then N is contained in A∩B, contradicting the factthat CoreG(A∩B)=1. We may assume without loss of generality that N∩A=N.

**(iv)** AN and BN are both supersoluble. Since N=(N∩A)(N∩B) and N=N∩A, it follows that N∩B is not contained in N∩A. Let n be any element of N∩B such that n/∈N∩A, and write N0 =〈n〉. Note that AN0 is a subgroup of G, and AN0∩N=(N∩A)N0. Therefore N0(N∩A) is a normal subgroup of AN0, and consequently A normalizes (A∩N)N0. This yields that A/CA(N/N∩A) acts as a power automorphism group on N/N∩A. This means that AN is supersoluble. If N∩B=N, then BN=B is supersoluble. On the contrary, if N∩B=N, we can argue as above and we obtain that BN is supersoluble. Consequently, ACG(N)/CG(N) and BCG(N)/CG(N) are both abelian groups of exponent dividing p−1. But then G/CG(N)=(ACG(N)/CG(N))(BCG(N)/CG(N)) is a π-group for some set of primes π such that if q∈π, then q divides p−1.

**(v)** Let B0 be a Hall π-subgroup of B. Then AB0∩N= A∩N.

This follows just by observing that AB0∩Nis contained in each Hall π′-subgroup of AB0 and every Hall π′-subgroup of A is a Hall π′-subgroup of AB0. Note that |G/CG(N)| is a π-number and AB0 contains a Hall π-subgroup of G. Therefore G=(AB0)CG(N). But then A∩N is a normal subgroup of G. The minimality of G yields either A∩N= 1or A∩N= N. This contradicts our assumption 1=N∩A=N, and so we cannot have N=(A∩N)(B∩N). Thus, by Lemma 3, we may assume N∩A=N∩B=1. Let M= CoreG(AN∩BN). Then N∩M= N and G/M is supersoluble by the minimality of G. Again, we reach a contradiction after several steps.

 **(vi)** M=N. Suppose that M=N. Since G/M is supersoluble, we know that N cannot be cyclic. Let us write C=CG(N), and consider the quotient group G/C. It is clear that G/C is supersoluble. Let Q/C=Op(G/C). Since Op(G/C)=1 and (G/C)′is nilpotent, it follows that Q/C is a normal Hall p′-subgroup of G/C. Let Bp′ be a Hall p′-subgroup of B. Since |N| divides |B:A∩B|, we have that (A∩B)Bp′ is a proper subgroup of B. Let T be a maximal subgroup of B containing (A∩B)Bp′. Then AT is a maximal subgroup of G and |G:AT|= p= |B:T|. If N is not contained in AT, we have G=(AT )N and AT∩N=1. Then |N|=p, a contradiction. Therefore N is contained in AT. In particular, the family S={X:X is a proper subgroup of B, (A∩B)Bp′X and NAX} is non-empty. Let R be an element of S of minimal order. Observe that AR has p-power index in G and thus ARC/C contains Op′(G/C). Regarding N as a AR-module over GF (p), we know, by Lemma 4, that N is a direct sum N=∏ti=1Ni, where Ni is an irreducible AR-module whose dimension is greater than 1, for all i∈{1,...,t}. Assume that (A∩B)Bp′=R. Then AR=ABp′ and thus N is contained in A, a contradiction. Therefore ABp′∩B=(A∩B)Bp′ is a proper subgroup of R. Let S be a maximal subgroup of R containing (A∩B)Bp′. From the minimality of R, we know that N is not contained in AS. Consequently, there exists some i∈{1,...,t} such that Ni is not contained in AS, which is a maximal subgroup of AR. Hence AR=(AS)Ni. Since Ni is a minimal normal subgroup of AR, it follows that AS∩Ni= 1and |Ni|= |AR:AS|= |R:S|= p, a contradiction.

**(vii)** M is an elementary abelian p-group. Note that M=N(M∩A)=N(M∩B) and |M∩A|=|M∩B|=|M|/|N|. Moreover, A(M∩B)is a subgroup of G such that A(M∩B)∩M= (M∩A)(M∩B). Hence (M∩A)(M∩B) is also a subgroup of G. If M∩A= M∩B, then M∩A is a normal subgroup of G contained in A∩B. This implies that M∩A=1 and consequently M=N, a contradiction. It yields that M∩A=M∩B. Next we see that (M∩A)(M∩B) is a normal subgroup of G. Since (M∩A)(M∩B)= M∩A(M∩B), we have that A normalizes (M∩A)(M∩B). Similarly, B normalises

 (M∩A)(M∩B) since (M∩A)(M∩B)= M∩B(M∩A). This implies normality of (M∩A)(M∩B) in G. Let X=(M∩A)(M∩B). Since we cannot have M∩A= M∩B, M∩A must be strictly contained in X. Thus X=X∩M=(X∩N)(M∩A) > M∩A gives us X∩N=1. But then X∩N=N, giving NX. Suppose that Q is a Hall p′-subgroup of M∩B. Then QA is a subgroup and so QA∩M=Q(M∩A) is also a subgroup which contains Q. Hence, as |M:M∩A|=pk for some k, we have that QM∩A∩B. Thus QB∩MM∩A∩B and similarly QA∩MM∩A∩B. Consequently, QM is contained in M∩A∩B. Since QM=Op(M), it follows that Op( M) is a normal subgroup of G contained in A∩B. Hence Op(M)=1, a contradiction, and consequently Q=1andMis a p-group. Hence N is contained in Z(M) and M=N×(M∩A)=N×(M∩B). Thus φ(M)=φ(M∩A)=φ(M∩B) is a normal subgroup of G contained in A∩B. This implies that φ(M)=1 and M is an elementary abelian p-group, as claimed.(viii) Final contradiction. We have from the previous steps that M∩A is not contained in M∩B and that M∩B is not contained in M∩A because otherwise, since |M∩A|=|M∩B|, it follows that M∩A=M∩B is a normal subgroup of G contained in A∩B. This would imply M∩A=M∩B=1, and M=(M∩A)N=N. This fact contradicts step (vi).

Let x be an element of M∩B such that x/∈M∩A. Then A〈x〉is a subgroup of G, and so is M0=A〈x〉∩M=(A∩M)〈x〉. Therefore M0 is an A-invariant subgroup of G. In particular, since M=(M∩A)(M∩B), we have that every subgroup of M/M∩A is A-invariant; that is, A/CA(M/M∩A) acts as a group of power automorphisms on M/M∩A. It is clear that M/M∩A is A-isomorphic to N. Consequently, A/CA(N) acts as a group of power automorphisms on N. This implies that A normalises each subgroup of N. A nalogously, B normalises each subgroup of N. It follows that N is a cyclic group. We argue as in step (ii) above to reach a final contradiction. We have that G/M is supersoluble and M is abelian. Therefore GUM and thus GU is abelian and complemented in G by a supersoluble normaliser, D say, by [8, Theorem V.5.18]. Since N is cyclic, we know that D covers N and thus NGU∩D=1, a contradiction. Proof of Theorem 2.

Let M=GU denote the supersoluble residual of G. Theorem 1yields that G/CoreG(A∩B) is supersoluble. Therefore M is contained in CoreG(A∩B). In particular, M is supersoluble. Let F(M) be the Fitting subgroup of M. Since A and Bare supersoluble, we have that [M,A]F(A)∩MF(M) and [M,B]F(B)∩MF(M). Consequently, [M,G] is contained in F(M). Note now that the chief factors of G between F(M) and Mare cyclic,and recall that G/M is supersoluble. Therefore we have that G/F (M) is supersoluble. This implies that M=F(M) and thus M is nilpotent. Consequently, G/F (G) is supersoluble. We now show that G/F (G) is metabelian. We prove first that A′ and B′ both centralise every chief factor of G. Let H/K be a chief factor of G. If H/K is cyclic, then as G′ centralizes H/K, so do A′ an dB′. Hence we may assume that H/K is a non-cyclic p-chief factor of G for some prime p. Note that we may assume that H is contained in M because G/M is supersoluble and H/K is non-cyclic. To simplify notation, we can consider K=1. Since F(G) centralizes H [8, Theorem A.13.8.b], G/CG(H ) is supersoluble. Let Ap′ be a Hall p′-subgroup of A. By Maschke’s theorem [8, Theorem A.11.5],H is a completely reducible Ap′-module and HAp′ is supersoluble because H is contained in A. Therefore Ap′/CAp′(H ) is abelian of exponent dividing p−1. This implies that the primes involved in |A/CA(H )| can only be p or divisors of p−1.The same is true for |B/CB(H )|. This implies that if p divides |G/CG(H )|, then p is the largest prime dividing |G/CG(H )|. But since Op(G/CG(H ))=1 and G/CG(H ) is supersoluble, it follows that G/CG(H ) must be a p′-group. Consider H as A-module over GF (p). Since ACG(H )/CG(H ) is a p′-group, we have that H is a completely reducible A-module and every irreducible A-submodule of H is cyclic. Consequently A′ centralizes H, and the same is true for B′. Let now U/V be a chief factor of G. Then G/CG(U/V )is the product of the abelian subgroups ACG(U/V )/CG(U/V ) and BCG(U/V )/CG(U/V ). By Itô’s theorem [9], we have that G/CG(U/V )is metabelian. Since F(G)is the intersection of the centralisers of all chief factors (again by [8, Theorem A.13.8.b]), we can conclude that G/F (G) is metabelian.3. Final remarks Finally, Theorem 1 enables us to give succinct proofs of earlier results on mutually permutable products.

**Corollary 1**[2, Theorem 3.2]. Let G=AB be the mutually permutable product of the subgroups A and B. If A is supersoluble and B is nilpotent, then G is supersoluble.

**Proof.** Assume that the assertion is false, and let G be a minimal counterexample. We have that G is a primitive group, and so G has a unique minimal normal subgroup, N say, with N=CG(N) a p-group for some prime p. Since G is not supersoluble, applying Theorem 1, we know that CoreG(A∩B)=1. This yields that N is contained in A∩B. Now, since N is contained in B, which is nilpotent, it follows that any p′-element of B must centralize N. Since CG(N)=N, we have that B itself is a p-group. Consequently, A must contain a Hall p′-subgroup of G. Now let T/N=Op′(G/N). The previous argument yields that T/N is contained in A/N. Note that if B=N, then G =AN= A is supersoluble, a contradiction. Thus N is a proper subgroup of B. This implies that p must divide |G:T|. Since G/N is supersoluble, p must divideq−1 for some prime q∈π(T/N). It is clear then that q can not divide p−1. Therefore there exists a Sylow q-subgroup Aq of A which centralizes N. Using that CG(N)=N, it yields that Aq=1 and thus q does not divide |G|,a contradiction.

**Corollary 2[2, Theorem 3.8].** Let G=AB be the mutually permutable product of thes upersoluble subgroups A and B. If G′ is nilpotent, then G is supersoluble. **Proof.** We assume the result to be false, and choose a minimal counterexample G. Thus G is a primitive group with unique minimal normal subgroup N. We also have that G=NM, where M is a maximal subgroup of G,N∩M=1 and N=F(G)=Op(G) for some prime p. Now G′ is nilpotent and thus G′=F(G)=N. Therefore M is an abelian group. Since N is self-centralising, arguing as we did in the previous corollary, we have that N is contained in A∩B. Note that M∼=G/N, and thus Op(M)=1. Since M is abelian, this yields that M is a p′-group. Thus M is in fact a Hall p′-subgroup of G. Applying [1, Theorem 1.3.2], wehave that there exist a Hall p′-subgroup Ap′ of A and a Hall p′-subgroup Bp′ of B suchthat M=Ap′Bp′. Since NA∩B, it follows that both Ap′ and Bp′ must have exponent dividing p−1.Regarding N as a M-module, it is easy to see that M must be a cyclic group. Now, since M=Ap′Bp′ has exponent dividing p−1, it follows that N is a cyclic group as well. This implies that G is supersoluble, a contradiction.

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