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On $\alpha^{\star}$-sets in Generalized Topological Spaces

Research Article

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Abstract: The aim of this paper is to introduce the new notion called μ-$\alpha^{\star}$-sets and study its properties in generalized topological spaces. Also we obtain some decompositions.

Keywords: μ-$\alpha^{\star}$-set, C_{μ}-set, B_{μ}-set.

1. Introduction

The theory of generalized topology was studied by A. Csàszàr [2] in 1997. In his papers properties of generalized topology, basic operators, generalized neighborhood system, some constructions for generalized topologies etc have been introduced and studied. It is well known that generalized topology in the sense of Csàszàr [2] is a generalization of topology on set. The aim of this paper is to introduce μ-$\alpha^{\star}$-sets and obtain some decompositions. Recall some generalized topological concepts which are very useful in the sequel.

Let X be a non-empty set and μ be a collection of subsets of X. Then μ is called generalized topology [2] (briefly GT) on X if $\phi \in \mu$ and $G_i \in \mu$ for i $\in$ I $\neq \phi$ implies G $= \bigcup_{i\in I} \mathrm{G}_i \in \mu$. We say μ is strong [4] if X $\in \mu$ and we call the pair (X, μ) a generalized topological space (briefly GTS). The elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets [2]. For A $\subseteq$ X, we denote by $c_{\mu}(A)$ the intersection of all μ-closed sets containing A and by i_{μ}(A) the union of all μ-open sets contained in A [5].

Remark 1.1 ([14]). *In a GTS (X, μ) the followings hold:*

1. $i_{\mu}(A \cap B) \subseteq i_{\mu}(A) \cap i_{\mu}(B)$;
2. $c_{\mu}(A \cup B) \supseteq c_{\mu}(A) \cup c_{\mu}(B)$.

Definition 1.2. *Let (X, μ) be a GTS and $A \subseteq X$. Then A is said to be*

1. *μ-semi-open [5] if $A \subseteq c_{\mu}(i_{\mu}(A))$.*

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2. *μ-preopen [5] if $A \subseteq i_\mu(c_\mu(A))$.*

3. *μ-α-open [5] if $A \subseteq i_\mu(c_\mu(i_\mu(A)))$.*

4. *μ-β-open [5] if $A \subseteq c_\mu(i_\mu(c_\mu(A)))$.*

5. *μr-open [8] if $A = i_\mu(c_\mu(A))$.*

The complement of a μ-semi-open (resp. μ-preopen, μ-α-open, μ-β-open, μr-open) set is called μ-semi-closed (resp. μ-preclosed, μ-α-closed, μ-β-closed, μr-closed) set. We denote by $\sigma(\mu)$ (resp. $\pi(\mu)$, $\alpha(\mu)$, $\beta(\mu)$) the classes of all μ-semi-open sets (resp. μ-preopen sets, μ-α-open sets, μ-β-open sets). Obviously, in [5], $\mu \subseteq \alpha(\mu) \subseteq \sigma(\mu) \subseteq \beta(\mu)$ and $\alpha(\mu) \subseteq \pi(\mu) \subseteq \beta(\mu)$. Clearly every μr-open set is μ-open but not conversely.

Definition 1.3 ([12])**.** *Let (X, μ) be a GTS and $A \subseteq X$. We denote by $c_\alpha(A)$ the intersection of all μ-α-closed sets containing A and by $i_\alpha(A)$ the union of all μ-α-open sets contained in A.*

Lemma 1.4 ([5])**.** *Let (X, μ) be a GTS and $A, B \subseteq X$, then the followings hold.*

1. *$i_\mu(A) \subseteq A \subseteq c_\mu(A)$;*

2. *$A \subseteq B$ implies $i_\mu(A) \subseteq i_\mu(B)$ and $c_\mu(A) \subseteq c_\mu(B)$;*

3. *$i_\mu(i_\mu(A)) = i_\mu(A)$ and $c_\mu(c_\mu(A)) = c_\mu(A)$;*

4. *$i_\mu(X - A) = X - c_\mu(A)$ and $c_\mu(X - A) = X - i_\mu(A)$;*

5. *$A \in \mu$ iff $A = i_\mu(A)$ and A is μ-closed iff $A = c_\mu(A)$.*

Definition 1.5 ([10])**.** *Let (X, μ) be a GTS and $A \subset X$. Then A is said to be μ-nowhere dense if $i_\mu(c_\mu(A)) = \phi$.*

Definition 1.6 ([6])**.** *In a GTS (X, μ), if μ is closed under finite intersections, then (X, μ) is called a quasi-topological space.*

Definition 1.7 ([11])**.** *Let (X, μ) be a quasi-topological space. For A and B of X the followings hold.*

1. *$i_\mu(A \cap B) = i_\mu(A) \cap i_\mu(B)$.*

2. *$c_\mu(A \cup B) = c_\mu(A) \cup c_\mu(B)$.*

Lemma 1.8 ([7])**.** *Let (X, μ) be a quasi-topological space. If $A \subset X$ and $U \in \mu$ then $U \cap c_\mu(A) \subset c_\mu(U \cap A)$.*

2. μ-$\alpha^\star$-sets

Definition 2.1. *A subset A of a GTS (X, μ) is said to be a μ-$\alpha^\star$-set if $i_\mu(A) = i_\mu(c_\mu(i_\mu(A)))$. We will denote the family of all μ-$\alpha^\star$-sets in a GTS (X, μ) by $\mu\alpha^\star(X)$.*

Remark 2.2. *The following two Examples show that μ-$\alpha^\star$-sets and μ-α-open sets are independent.*

Example 2.3. *Let $X = R$ be the set of real numbers and $\mu = \{\phi, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. Let $A \subseteq X$ and $A \neq \phi$ be such that $A \cap \{1, 2, 3\} = \phi$. Then $i_\mu(A) = \phi$ and $i_\mu(c_\mu(i_\mu(A))) = \phi$. Then A is a μ-$\alpha^\star$-set but not μ-α-open.*

Example 2.4. *Let $X = \{a, b, c, d\}$ and $\mu = \{\phi, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Let $A = \{a, b\}$. Then $i_\mu(c_\mu(i_\mu(A))) = \{a, b, c\} \supseteq A$ and so A is μ-α-open but it is not a μ-$\alpha^\star$-set.*

Theorem 2.5. *Let (X, μ) be a GTS and $A \subseteq X$. Then the following are equivalent.*

1. *A is a μ-$\alpha^\star$-set;*
2. *$X - A$ is μ-β-open;*
3. *$i_\mu(A)$ is μr-open.*

Proof. (1) $\Rightarrow$ (2): Suppose A is a μ-$\alpha^\star$-set. Claim $X - A$ is μ-β-open. Now $c_\mu(i_\mu(c_\mu(X - A))) = X - i_\mu(c_\mu(i_\mu(A)))) = X - i_\mu(A) = c_\mu(X - A) \supseteq X - A$. So $X - A$ is μ-β-open.

(2) $\Rightarrow$ (3): Suppose $X - A$ is a μ-β-open set. Claim $i_\mu(A)$ is μr-open. Since $X - A \in \beta(\mu)$, $X - A \subseteq c_\mu(i_\mu(c_\mu(X - A)))$ $= X - i_\mu(c_\mu(i_\mu(A)))$ which implies that $i_\mu(c_\mu(i_\mu(A))) \subseteq A$ and $i_\mu(c_\mu(i_\mu(A))) \subseteq i_\mu(A)$. But always $i_\mu(A) \subseteq i_\mu(c_\mu(i_\mu(A)))$ and so $i_\mu(A) = i_\mu(c_\mu(i_\mu(A)))$. Then $i_\mu(A)$ is μr-open.

(3) $\Rightarrow$ (1): is clear. □

Theorem 2.6. *Let (X, μ) be a GTS and $A \subseteq X$, then A is μ-$\alpha^\star$-set and μ-α-open set if and only if it is μr-open.*

Proof. Suppose $A \subseteq X$ is both μ-$\alpha^\star$-set and μ-α-open. Then $i_\mu(A) = i_\mu(c_\mu(i_\mu(A)))$ and A $\subseteq i_\mu(c_\mu(i_\mu(A)))$. We have $A \subseteq i_\mu(A)$ and so i_μ(A) $= A$. By Theorem 2.5, A is μr-open.

Conversely, suppose A is μr-open. Hence A is μ-open and hence μ-α-open. Also $A = i_\mu(c_\mu(A))$ implies that $i_\mu(A) = i_\mu(c_\mu(i_\mu(A)))$. Hence A is a μ-$\alpha^\star$-set. □

Theorem 2.7. *Let (X, μ) be a GTS and $A \subseteq X$ is a μ-semi-closed set, then A is μ-$\alpha^\star$-set.*

Proof. Let A be a μ-semi-closed set of X. Since $i_\mu(A) \subseteq A$, $c_\mu(i_\mu(A)) \subseteq c_\mu(A)$ and $i_\mu(c_\mu(i_\mu(A))) \subseteq i_\mu(c_\mu(A)) \subseteq A$ implies $i_\mu(c_\mu(i_\mu(A))) \subseteq i_\mu(A)$. Always $i_\mu(A) \subseteq i_\mu(c_\mu(i_\mu(A)))$. Hence A is μ-$\alpha^\star$-set. □

Remark 2.8. *The converse of the above Theorem need not be true.*

Example 2.9. *Let $X = \{a, b, c, d\}$ and $\mu = \{\phi, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Let $A = \{a, c\}$ then $i_\mu(c_\mu(A)) = \{a, b, c\}$. Since $i_\mu(c_\mu(A)) \subseteq A$, A is not μ-semi-closed. But $i_\mu(c_\mu(i_\mu(A))) = \{a\} = i_\mu(A)$ and so A is a μ-$\alpha^\star$-set.*

Theorem 2.10. *Let (X, μ) be a GTS and $A \subseteq X$. Then the following are equivalent.*

1. *A is μ-semi-closed and μ-α-open.*
2. *A is μ-$\alpha^\star$-set and μ-α-open.*
3. *$X - A$ is μ-β-open and A is μ-α-open.*
4. *A is μr-open.*
5. *A is μ-semi-closed and μ-preopen.*

Proof. (1) $\Rightarrow$ (2) follows from Theorem 2.7.

(2) $\Rightarrow$ (3) follows from Theorem 2.5.

(2) $\Leftrightarrow$ (4) follows from Theorem 2.6.

(4) $\Rightarrow$ (1) and (4) $\Leftrightarrow$ (5) are clear. □

Remark 2.11. *The union of two μ-$\alpha^\star$-sets need not be a μ-$\alpha^\star$-set.*

Example 2.12. *In Example 2.9, if $A = \{a\}$ and $B = \{b\}$, then A and B are μ-$\alpha^{\star}$-sets but their union $A \cup B = \{a,\ b\}$ is not a μ-$\alpha^{\star}$-set.*

Definition 2.13. *A subset A of a GTS $(X,\ \mu)$ is said to be a C_{μ}-set (resp. B_{μ}-set) if there exists $U \in \mu$ and $D \in \mu\alpha^{\star}(X)$ (resp. D is μ-semi-closed) such that $A = U \cap D$.*
The family of all C_{μ}-sets (resp. B_{μ}-sets) in $(X,\ \mu)$ is denoted by $C_{\mu}(X)$ (resp. $B_{\mu}(X)$).

Theorem 2.14. *Let $(X,\ \mu)$ be a GTS and $A \subseteq X$. Then the following hold.*

1. *$B_{\mu}(X) \subseteq C_{\mu}(X)$.*
2. *$\mu \subseteq B_{\mu}(X)$.*
3. *If $X \in \mu$ then $\mu\alpha^{\star}(X) \subseteq C_{\mu}(X)$.*
4. *If $X \in \mu$ and $A \subseteq X$ is μ-semi-closed then $A \in B_{\mu}(X)$.*

Proof. (1) The proof follows from Theorem 2.7.
(2) Consider $A \in \mu$. Then $A = A \cap X$ where $A \in \mu$ and X is μ-semi-closed. Clearly μ-open set is B_{μ}-set.
The proof of (3) and (4) are clear. □

Remark 2.15. *In Theorem 2.14, the separate converses of (1) and (2) are not true as shown by the following Example.*

Example 2.16. *Consider the GTS in Example 2.4.*

1. *It is clear that $C_{\mu}(X) = \{\phi,\ \{a\},\ \{b\},\ \{c\},\ \{a,\ b\},\ \{b,\ c\},\ \{a,\ b,\ c\}\}$ and $B_{\mu}(X) = \{\phi,\ \{a\},\ \{c\},\ \{a,\ b\},\ \{b,\ c\},\ \{a,\ b,\ c\}\}$. Thus $\{b\} \in C_{\mu}(X)$ but $\{b\} \notin B_{\mu}(X)$.*
2. *Clearly $\{a\} \in B_{\mu}(X)$ but $\{a\} \notin \mu$.*

Remark 2.17. *In Theorem 2.14(3) and 2.14(4), the condition $X \in \mu$ cannot be dropped as shown by the following Example.*

Example 2.18. *Let $X = \{a,\ b,\ c,\ d\}$. If we take μ not containing X where $\mu = \{\phi,\ \{a,\ c\},\ \{b,\ c\},\ \{a,\ b,\ c\}\}$. Then*

1. *$\mu\alpha^{\star}$-sets are ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a,\ b\}$, $\{a,\ d\}$, $\{b,\ d\}$,$\{c,\ d\}$, $\{a,\ b,\ c\}$, $\{a,\ b,\ d\}$, X and C_{μ}-sets are ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,\ b\}$, $\{b,\ c\}$, $\{a,\ b,\ c\}$. We obtain that $\mu\alpha^{\star}(X) \nsubseteq C_{\mu}(X)$.*
2. *μ-semi-closed sets are ϕ, $\{a\}$, $\{b\}$, $\{d\}$, $\{a,\ d\}$, $\{b,\ d\}$, X and B_{μ}-sets are ϕ, $\{a\}$, $\{b\}$, $\{a,\ c\}$, $\{b,\ c\}$, $\{a,\ b,\ c\}$. We obtain that $\{a,\ d\}$ is μ-semi-closed but not B_{μ}-set.*

Theorem 2.19. *Let $(X,\ \mu)$ be a GTS and $A \subseteq X$. Then the followings hold.*

1. *If $A \in \mu$, then A is both μ-α-open and C_{μ}-set.*
2. *If $(X,\ \mu)$ is a quasi-topological space, then the converse of (1) holds.*

Proof. (1) If $A \in \mu$, then clearly A is μ-α-open and by *Theorem* 2.14(2), $A \in B_{\mu}(X)$ which implies that $A \in C_{\mu}(X)$.
(2) Suppose A is μ-α-open set and a C_{μ}-set. Then $A \subseteq i_{\mu}(c_{\mu}(i_{\mu}(A)))$ and there exists $U \in \mu$ and D is a μ-$\alpha^{\star}$-set such that $A = U \cap D$. Then $A \subseteq i_{\mu}(c_{\mu}(i_{\mu}(A))) \subseteq i_{\mu}(c_{\mu}(i_{\mu}(U \cap D)) \subseteq i_{\mu}(c_{\mu}(i_{\mu}(D))) = i_{\mu}(D)$. Therefore $A = U \cap A \subseteq U \cap i_{\mu}(D) \subseteq U \cap D = A$ which implies that $A = U \cap i_{\mu}(D)$. We know that $i_{\mu}(D)$ is μ-open for any subset $D \subseteq X$. Since μ is a quasi topology, $U \cap i_{\mu}(D) = A \in \mu$. □

Remark 2.20. *In the Theorem 2.19(2) the condition of quasi topological space (X, μ) cannot be dropped as shown by the following Example.*

Example 2.21. *Let $X = \{a, b, c, d, e\}$ and $\mu = \{\phi, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$. It is clear that $\alpha(\mu) = \{\phi, \{a\}, \{d\}, \{a, b\}, \{a, c\} \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$. Also $C_\mu(X) = \{\phi, \{a\}, \{b\},\{c\}, \{d\}, \{a, b\}, \{a, c\} \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\} \{a, b, c, d\}\}$. It is observe that $\{a, b\}$ is both μ-α-open and C_μ-set but $\{a, b\} \notin \mu$.*

Remark 2.22. *In a GTS (X, μ), the following hold: $c_\mu(A \cap B) \subseteq c_\mu(A) \cap c_\mu(B)$.*

Theorem 2.23. *Let (X, μ) be a quasi topological space and $A \subseteq X$. Then A is μ-α-open if and only if $A = U - B$ where $U \in \mu$ and B is μ-nowhere dense.*

Proof. Suppose A is μ-α-open. Then $A \subseteq i_\mu(c_\mu(i_\mu(A))) = B$, say. Claim A can be expressed as difference of μ-open sets and μ-nowhere dense sets. Now $(i_\mu(c_\mu(B - A)) = i_\mu(c_\mu(B \cap (X - A))) \subseteq (i_\mu(c_\mu(B))) \cap i_\mu(c_\mu(X - A)) = i_\mu(c_\mu(i_\mu(A))) \cap (X - c_\mu(i_\mu(A)) \subseteq i_\mu(c_\mu(i_\mu(A))) \cap (X - i_\mu(c_\mu(i_\mu(A))) = \phi$ and so $B - A$ is μ-nowhere dense. Since μ is a GT, B $\in \mu$. Therefore $A = B - (B - A)$ where $B \in \mu$ and $B - A$ is μ-nowhere dense.
Conversely, suppose $A = U - B$ where $U \in \mu$ and B is μ-nowhere dense. Claim A is μ-α-open. Now $i_\mu(c_\mu(i_\mu(A))) = i_\mu(c_\mu(i_\mu(U - B))) = i_\mu(c_\mu(i_\mu(U \cap X - B))) = i_\mu(c_\mu(U \cap i_\mu(X - B))) \supseteq i_\mu(U \cap c_\mu(i_\mu(X - B)))$, by Lemma 1.8 and so $i_\mu(c_\mu(i_\mu(A))) \supseteq i_\mu(U \cap (X - i_\mu(c_\mu(B)))) = i_\mu(U \cap X) = i_\mu(U) = U \supseteq U - B = A$. Therefore A is μ-α-open. □

References

[1] S.Z.Bai and Y.P.Zuo, *On g-α-irresolute functions*, Acta Math. Hungar., 130(4)(2011), 382-389.

[2] A.Csàszàr, *Generalized open sets*, Acta Math. Hungar., 75(1-2)(1997), 65-87.

[3] A.Csàszàr, *Generalized topology, generalized continuity*, Acta Math. Hungar., 96(2002), 351-357.

[4] A.Csàszàr, *Extremally disconnected generalized topologies*, Annales Univ. Sci. Budapest., 47(2004), 91-96.

[5] A.Csàszàr, *Generalized open sets in generalized topologies*, Acta Math.Hungar., 106(2005), 53-66.

[6] A.Csàszàr, *Further remarks on the formula for γ-interior*, Acta Math.Hungar., 113(2006), 325-332.

[7] A.Csàszàr, *Remarks on quasi-topologies*, Acta Math. Hungar., 119(2007), 197-200.

[8] A.Csàszàr, *δ- and θ- modifications of generalized topologies*, Acta Math Hungar., 120(3)(2008), 275-279.

[9] A.Csàszàr, *Weak Structures*, Acta Math. Hungar., 131(1-2)(2011), 193-195.

[10] E.Ekici, *Generalized hyperconnectedness*, Acta Math. Hungar., 133(1-2)(2011), 140-147.

[11] R.Jamunarani and P.Jeyanthi, *Reguler sets in generalized topological spaces*, Acta Math. Hungar., 135(4)(2012), 342-349.

[12] W.K.Min, *Generalized continuous functions defined by generalized open sets on generalized topological spaces*, Acta Math. Hungar., 128(4) (2010), 299-306.

[13] S.Thamaraiselvi and M.Navaneethakrishnan, *On $\alpha^\star$-sets and a decomposition theorem*, Jordan Journal of Mathematics and Statistics., 6(1)(2013), 29-36.

[14] GE Xun and GE Ying, *μ-Separations in generalized topological spaces*, Appl. Math. J. Chinese Univ., 25(2)(2010), 243-252.