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Abstract: In this paper, another generalized class of τ^* called weakly $\mathcal{I}_g$ - ω -closed sets is studied and the notion of weakly $\mathcal{I}_g$ - ω -open sets in ideal topological spaces is also studied. The relationships of weakly $\mathcal{I}_g$ - ω -closed sets with various other sets are investigated.

MSC: 54A05, 54A10.

Keywords: τ^* , generalized class, weakly $\mathcal{I}_g$ - ω -closed set, ideal topological space, generalized closed set, $\mathcal{I}_g$ - ω -closed set, $pre_{\mathcal{I}}^*$ -closed set, $pre_{\mathcal{I}}^*$ -open set.

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1. Introduction

The first step of generalizing closed sets (briefly, g -closed sets) was done by Levine in 1970 [13]. He defined a subset S of a topological space (X, τ) to be g -closed if its closure is contained in every open superset of S . As the weak form of g -closed sets, the notion of weakly g -closed sets was introduced and studied by Sundaram and Nagaveni [18]. Sundaram and Pushpalatha [19] introduced and studied the notion of strongly g -closed sets, which are weaker than closed sets and stronger than g -closed sets. Park and Park [17] introduced and studied the notion of mildly g -closed sets, which is properly placed between the class of strongly g -closed sets and the class of weakly g -closed sets. Moreover, the relations with other notions directly or indirectly connected with g -closed sets were investigated by them. In 1999, Dontchev et al. [6] studied the notion of generalized closed sets in ideal topological spaces called $\mathcal{I}_g$ -closed sets. In 2008, Navaneethakrishnan and Paulraj Joseph studied some characterizations of normal spaces via $\mathcal{I}_g$ -open sets [14]. In 2013, Ekici and Ozen [8] introduced a generalized class of τ^* in ideal topological spaces. The notion of ω -open sets in topological spaces introduced by Hdeib [9] has been studied in recent years by a good number of researchers like Noiri et al [16], Al-Omari and Noorani [3, 4] and Khalid Y. Al-Zoubi [11]. The main aim of this paper is to study another generalized class of τ^* called weakly $\mathcal{I}_g$ - ω -open sets in ideal topological spaces. Moreover, this generalized class of τ^* generalize $\mathcal{I}_g$ - ω -open sets and weakly $\mathcal{I}_g$ - ω -open sets. The relationships of weakly $\mathcal{I}_g$ - ω -closed sets with various other sets are discussed.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{Q}$, $(\mathbb{R} - \mathbb{Q})$, $(\mathbb{R} - \mathbb{Q})_-$ and $(\mathbb{R} - \mathbb{Q})_+$) denotes the set of real numbers (resp. the set of rational numbers, the set of irrational numbers, the set of negative irrational numbers and the set of positive irrational

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numbers). In this paper, (X, τ) represents a topological space on which no separation axioms are assumed unless explicitly stated. The closure and interior of a subset G of a topological space (X, τ) will be denoted by $cl(G)$ and $int(G)$, respectively.

Definition 2.1. A subset G of a topological space (X, τ) is said to be

- (1). g -closed [13] if $cl(G) \subseteq H$ whenever $G \subseteq H$ and H is open in X ;
- (2). g -open [13] if $X \setminus G$ is g -closed;
- (3). weakly g -closed [18] if $cl(int(G)) \subseteq H$ whenever $G \subseteq H$ and H is open in X ;
- (4). strongly g -closed [19] if $cl(G) \subseteq H$ whenever $G \subseteq H$ and H is g -open in X .

Definition 2.2 ([21]). In a topological space (X, τ) , a point p in X is called a condensation point of a subset H if for each open set U containing p , $U \cap H$ is uncountable.

Definition 2.3 ([9]). A subset H of a topological space (X, τ) is called ω -closed if it contains all its condensation points. The complement of an ω -closed set is called ω -open.

It is well known that a subset W of a topological space (X, τ) is ω -open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω -open sets, denoted by τ_ω , is a topology on X , which is finer than τ . The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.

Lemma 2.4 ([9]). Let H be a subset of a topological space (X, τ) . Then

- (1). H is ω -closed in X if and only if $H = cl_\omega(H)$.
- (2). $cl_\omega(X \setminus H) = X \setminus int_\omega(H)$.
- (3). $cl_\omega(H)$ is ω -closed in X .
- (4). $x \in cl_\omega(H)$ if and only if $H \cap G \neq \emptyset$ for each ω -open set G containing x .
- (5). $cl_\omega(H) \subseteq cl(H)$.
- (6). $int(H) \subseteq int_\omega(H)$.

Lemma 2.5 ([11]). If A is an ω -open subset of a space (X, τ) , then $A - C$ is ω -open for every countable subset C of X .

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

- (1). $A \in \mathcal{I}$ and $B \subseteq A$ imply $B \in \mathcal{I}$ and
- (2). $A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$ [12].

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X , if $\mathbb{P}(X)$ is the set of all subsets of X , a set operator $(.)^* : \mathbb{P}(X) \rightarrow \mathbb{P}(X)$, called a local function [12] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the $\star$ -topology and finer than τ , is defined by $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [20]. We will simply write A^* for $A^*(\mathcal{I}, \tau)$ and τ^* for $\tau^*(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X , then $(X, \tau, \mathcal{I})$ is called an ideal topological space. On the other hand, $(A, \tau_A, \mathcal{I}_A)$ where τ_A is the relative topology on A and $\mathcal{I}_A = \{A \cap J : J \in \mathcal{I}\}$ is an ideal topological subspace for the ideal topological space $(X, \tau, \mathcal{I})$ and $A \subseteq X$ [10]. For a subset $A \subseteq X$, $cl^*(A)$ and $int^*(A)$ will, respectively, denote the closure and the interior of A in (X, τ^*) .

Proposition 2.6 ([1]). Let $(X, \tau, \mathcal{I})$ be an ideal topological space and H a subset of X . If $\mathcal{I} = \{\phi\}$ (resp. $\mathbb{P}(X)$), then $H^* = cl(H)$ (resp. ϕ).

Lemma 2.7 ([10]). Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B subsets of X . Then the following properties hold:

- (1). $A \subseteq B \Rightarrow A^* \subseteq B^*$,
- (2). $A^* = cl(A^*) \subseteq cl(A)$,
- (3). $(A^*)^* \subseteq A^*$,
- (4). $(A \cup B)^* = A^* \cup B^*$,
- (5). $(A \cap B)^* \subseteq A^* \cap B^*$.

Definition 2.8. A subset G of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be

- (1). $\mathcal{I}_g$ -closed [6, 14] if $G^* \subseteq H$ whenever $G \subseteq H$ and H is open in $(X, \tau, \mathcal{I})$.
- (2). $pre_{\mathcal{I}}^*$ -open [7] if $G \subseteq int^*(cl(G))$.
- (3). $pre_{\mathcal{I}}^*$ -closed [7] if $X \setminus G$ is $pre_{\mathcal{I}}^*$ -open (or) $cl^*(int(G)) \subseteq G$.
- (4). $\mathcal{I}_g$ - ω -closed [5] if $G^* \subseteq H$ whenever $G \subseteq H$ and H is ω -open in $(X, \tau, \mathcal{I})$.
- (5). $\mathcal{I}$ -R closed [2] if $G = cl^*(int(G))$.
- (6). $\star$ -closed [10] if $G = cl^*(G)$ or $G^* \subseteq G$.

Remark 2.9 ([8]). In any ideal topological space, every $\mathcal{I}$ -R closed set is $\star$ -closed but not conversely.

Definition 2.10 ([15]). In an ideal topological space $(X, \tau, \mathcal{I})$, a subset G of X is said to be weakly $\mathcal{I}_g$ - ω -closed if $(int(G))^* \subseteq H$ whenever $G \subseteq H$ and H is ω -open in X .

Example 2.11 ([15]). In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$ and $\mathcal{I} = \{\phi\}$,

- (1). For $G = \mathbb{R} - \mathbb{Q}$, if H is any ω -open subset of $\mathbb{R}$ such that $G \subseteq H$, then $(int(G))^* = \phi^* = \phi \subseteq H$ and hence G is weakly $\mathcal{I}_g$ - ω -closed in X .
- (2). $K = \mathbb{Q} \subseteq \mathbb{Q}$, $\mathbb{Q}$ being ω -open whereas $(int(\mathbb{Q}))^* = \mathbb{Q}^* = cl(\mathbb{Q}) = \mathbb{R} \not\subseteq \mathbb{Q}$ which implies $K = \mathbb{Q}$ is not weakly $\mathcal{I}_g$ - ω -closed in X .

Definition 2.12 ([15]). A subset G in an ideal topological space $(X, \tau, \mathcal{I})$ is said to be weakly $\mathcal{I}_g$ - ω -open if $X \setminus G$ is weakly $\mathcal{I}_g$ - ω -closed.

Theorem 2.13 ([15]). In an ideal topological space $(X, \tau, \mathcal{I})$, a subset G of X is weakly $\mathcal{I}_g$ - ω -closed $\Leftrightarrow (int(G))^* \subseteq G$.

Proposition 2.14 ([15]). In an ideal topological space $(X, \tau, \mathcal{I})$, every $\mathcal{I}_g$ - ω -closed set is weakly $\mathcal{I}_g$ - ω -closed but not conversely.

Theorem 2.15 ([5]). In an ideal topological space $(X, \tau, \mathcal{I})$, a subset G is $\star$ -closed if and only if it is $\mathcal{I}_g$ - ω -closed.

Theorem 2.16 ([5]). In an ideal topological space $(X, \tau, \mathcal{I})$, every $\mathcal{I}_g$ - ω -closed set is $\mathcal{I}_g$ -closed but not conversely.

3. Properties of Weakly $\mathcal{I}_g$ - ω -closed Sets

Theorem 3.1. *In an ideal topological space $(X, \tau, \mathcal{I})$, for a subset G of X , the following properties are equivalent.*

- (1). G is weakly $\mathcal{I}_g$ - ω -closed;
- (2). $(\text{int}(G))^* \setminus G = \phi$;
- (3). $\text{cl}^*(\text{int}(G)) \setminus G = \phi$;
- (4). $\text{cl}^*(\text{int}(G)) \subseteq G$;
- (5). G is $\text{pre}_{\mathcal{I}}^*$ -closed.

Proof.

- (1) $\Leftrightarrow$ (2) G is weakly $\mathcal{I}_g$ - ω -closed $\Leftrightarrow (\text{int}(G))^* \subseteq G$ by Theorem 2.13 $\Leftrightarrow (\text{int}(G))^* \setminus G = \phi$.
- (2) $\Leftrightarrow$ (3) $(\text{int}(G))^* \setminus G = \phi \Leftrightarrow \text{cl}^*(\text{int}(G)) \setminus G = ((\text{int}(G))^* \cup \text{int}(G)) \setminus G = [(\text{int}(G))^* \setminus G] \cup [\text{int}(G) \setminus G] = \text{int}(G) \setminus G = \phi$.
- (3) $\Leftrightarrow$ (4) $\text{cl}^*(\text{int}(G)) \setminus G = \phi \Leftrightarrow \text{cl}^*(\text{int}(G)) \subseteq G$.
- (4) $\Leftrightarrow$ (5) $\text{cl}^*(\text{int}(G)) \subseteq G \Leftrightarrow G$ is $\text{pre}_{\mathcal{I}}^*$ -closed by (3) of Definition 2.8. □

Theorem 3.2. *In an ideal topological space $(X, \tau, \mathcal{I})$, if G is weakly $\mathcal{I}_g$ - ω -closed, then $G \cup (X - (\text{int}(G))^*)$ is weakly $\mathcal{I}_g$ - ω -closed.*

Proof. Since G is weakly $\mathcal{I}_g$ - ω -closed, $(\text{int}(G))^* \subseteq G$ by Theorem 2.13. Then $X - G \subseteq X - (\text{int}(G))^*$ and $G \cup (X - G) \subseteq G \cup (X - (\text{int}(G))^*)$. Thus $X \subseteq G \cup (X - (\text{int}(G))^*)$ and so $G \cup (X - (\text{int}(G))^*) = X$. Hence $G \cup (X - (\text{int}(G))^*)$ is weakly $\mathcal{I}_g$ - ω -closed. □

Theorem 3.3. *In an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

- (1). G is a $\star$ -closed and an open set,
- (2). G is a $\mathcal{I}$ - R closed and an open set,
- (3). G is a weakly $\mathcal{I}_g$ - ω -closed and an open set.

Proof.

- (1) $\Rightarrow$ (2): Since G is $\star$ -closed and open, $G = \text{cl}^*(G)$ and $G = \text{int}(G)$. Thus $G = \text{cl}^*(\text{int}(G))$ and $G = \text{int}(G)$. Hence G is $\mathcal{I}$ - R closed and open.
- (2) $\Rightarrow$ (3): Since G is $\mathcal{I}$ - R closed and open, $G = \text{cl}^*(\text{int}(G)) = (\text{int}(G))^* \cup \text{int}(G) = (\text{int}(G))^* \cup G$. Thus $(\text{int}(G))^* \subseteq G$. By Theorem 2.13, G is weakly $\mathcal{I}_g$ - ω -closed and open.
- (3) $\Rightarrow$ (1): Since G is weakly $\mathcal{I}_g$ - ω -closed, $(\text{int}(G))^* \subseteq G$ by Theorem 2.13. Again G is open implies $G^* = (\text{int}(G))^* \subseteq G$. Thus G is $\star$ -closed and open. □

Theorem 3.4. *In an ideal topological space $(X, \tau, \mathcal{I})$, every closed set is weakly $\mathcal{I}_g$ - ω -closed.*

Proof. If A is closed, then A is $\star$ -closed and $\mathcal{I}_g$ - ω -closed by Theorem 2.15. By Proposition 2.14, A is weakly $\mathcal{I}_g$ - ω -closed. □

Remark 3.5. *The converse of Theorem 3.4 is not true follows from the following example.*

Example 3.6. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \{1\}\}$ and the ideal $\mathcal{I} = \mathbb{P}(\mathbb{R})$, for $G = \{1\}$, if H is any ω -open subset of $\mathbb{R}$ such that $G \subseteq H$, then $(\text{int}(G))^* = G^* = \phi \subseteq H$ and hence G is weakly $\mathcal{I}_g$ - ω -closed. But G is not closed for $\text{cl}(G) = \mathbb{R} \not\subseteq G$.*

Remark 3.7. The following example shows that the concepts of $\mathcal{I}_g$ -closedness and weakly $\mathcal{I}_g$ - ω -closedness are independent of each other.

Example 3.8. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and the ideal $\mathcal{I} = \{\phi\}$,

- (1). for $G = (\mathbb{R} - \mathbb{Q})_+ =$ the set of positive irrationals, $\text{int}(G) = \phi$. So $(\text{int}(G))^* = \phi^* = \phi \subseteq G$ and thus G is weakly $\mathcal{I}_g$ - ω -closed by Theorem 2.13. But G is not $\mathcal{I}_g$ -closed for $G \subseteq (\mathbb{R} - \mathbb{Q}) \in \tau$ whereas $G^* = \text{cl}(G) = \mathbb{R} \not\subseteq \mathbb{R} - \mathbb{Q}$.
- (2). for $G = (\mathbb{R} - \mathbb{Q}) \cup \{1\}$, $\mathbb{R}$ is the only open set containing G . Hence G is $\mathcal{I}_g$ -closed. But G is not weakly $\mathcal{I}_g$ - ω -closed for $(\text{int}(G))^* = (\mathbb{R} - \mathbb{Q})^* = \text{cl}(\mathbb{R} - \mathbb{Q}) = \mathbb{R} \not\subseteq G$.

Remark 3.9. In an ideal topological space $(X, \tau, \mathcal{I})$, the following relations hold for a subset G of X .

$$\begin{array}{ccc}
 \text{closed} & \longrightarrow & \text{weakly } \mathcal{I}_g\text{-}\omega\text{-closed} \\
 \downarrow & & \uparrow \\
 \star\text{-closed} & \longleftrightarrow & \mathcal{I}_g\text{-}\omega\text{-closed} \longrightarrow \mathcal{I}_g\text{-closed}
 \end{array}$$

Where $A \longleftrightarrow B$ means A implies and is implied by B and $A \rightarrow B$ means A implies B but not conversely.

Theorem 3.10. In an ideal topological space $(X, \tau, \mathcal{I})$, A^* is always weakly $\mathcal{I}_g$ - ω -closed for every subset A of X .

Proof. Since $(A^*)^* \subseteq A^*$ [10], A^* is $\star$ -closed. Hence A^* is $\mathcal{I}_g$ - ω -closed by Theorem 2.15 and weakly $\mathcal{I}_g$ - ω -closed by Proposition 2.14. $\square$

4. Further Properties

Theorem 4.1. In an ideal topological space $(X, \tau, \mathcal{I})$, if G is weakly $\mathcal{I}_g$ - ω -closed and H is a subset such that $G \subseteq H \subseteq \text{cl}^*(\text{int}(G))$, then H is weakly $\mathcal{I}_g$ - ω -closed.

Proof. Since G is weakly $\mathcal{I}_g$ - ω -closed, $\text{cl}^*(\text{int}(G)) \subseteq G$ by (4) of Theorem 3.1. Thus by assumption, $G \subseteq H \subseteq \text{cl}^*(\text{int}(G)) \subseteq G$. Then $G = H$ and so H is weakly $\mathcal{I}_g$ - ω -closed. $\square$

Corollary 4.2. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If G is a weakly $\mathcal{I}_g$ - ω -closed set and an open set, then $\text{cl}^*(G)$ is weakly $\mathcal{I}_g$ - ω -closed.

Proof. Since G is open in X , $G \subseteq \text{cl}^*(G) = \text{cl}^*(\text{int}(G))$. G is weakly $\mathcal{I}_g$ - ω -closed implies $\text{cl}^*(G)$ is weakly $\mathcal{I}_g$ - ω -closed by Theorem 4.1. $\square$

Theorem 4.3. In an ideal topological space $(X, \tau, \mathcal{I})$, a nowhere dense subset is weakly $\mathcal{I}_g$ - ω -closed.

Proof. If G is a nowhere dense subset in X then $\text{int}(\text{cl}(G)) = \phi$. Since $\text{int}(G) \subseteq \text{int}(\text{cl}(G))$, $\text{int}(G) = \phi$. Hence $(\text{int}(G))^* = \phi^* = \phi \subseteq G$. Thus, G is weakly $\mathcal{I}_g$ - ω -closed in $(X, \tau, \mathcal{I})$ by Theorem 2.13. $\square$

Remark 4.4. The converse of Theorem 4.3 is not true in general as shown in the following example.

Example 4.5. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and $\mathcal{I} = \mathbb{P}(\mathbb{R})$, for $G = \mathbb{R} - \mathbb{Q}$, $(\text{int}(G))^* = G^* = \phi \subseteq G$. Hence $G = \mathbb{R} - \mathbb{Q}$ is weakly $\mathcal{I}_g$ - ω -closed by Theorem 2.13. On the other hand, $\text{int}(\text{cl}(G)) = \text{int}(\mathbb{R}) = \mathbb{R} \neq \phi$ and thus $G = \mathbb{R} - \mathbb{Q}$ is not nowhere dense in X .

Remark 4.6. In an ideal topological space $(X, \tau, \mathcal{I})$, the intersection of two weakly $\mathcal{I}_g$ - ω -closed subsets is weakly $\mathcal{I}_g$ - ω -closed.

Proof. Let A and B be weakly $\mathcal{I}_g$ - ω -closed subsets in $(X, \tau, \mathcal{I})$. Then $(\text{int}(A))^* \subseteq A$ and $(\text{int}(B))^* \subseteq B$ by Theorem 2.13. Also $[\text{int}(A \cap B)]^* \subseteq [\text{int}(A)]^* \cap [\text{int}(B)]^* \subseteq A \cap B$. This implies that $A \cap B$ is weakly $\mathcal{I}_g$ - ω -closed by Theorem 2.13. $\square$

Remark 4.7. In an ideal topological space $(X, \tau, \mathcal{I})$, the union of two weakly $\mathcal{I}_g$ - ω -closed subsets need not be weakly $\mathcal{I}_g$ - ω -closed.

Example 4.8. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and ideal $\mathcal{I} = \{\phi\}$, for $A = (\mathbb{R} - \mathbb{Q})_+ =$ the set of positive irrationals and $B = (\mathbb{R} - \mathbb{Q})_- =$ the set of negative irrationals, $\text{int}(A) = \phi$ and $\text{int}(B) = \phi$ respectively. So $(\text{int}(A))^* = \phi^* = \phi \subseteq A$ and thus A is weakly $\mathcal{I}_g$ - ω -closed by Theorem 2.13. Similarly B is also weakly $\mathcal{I}_g$ - ω -closed. But $\text{int}(A \cup B) = \text{int}(\mathbb{R} - \mathbb{Q}) = \mathbb{R} - \mathbb{Q}$. So $[\text{int}(A \cup B)]^* = (\mathbb{R} - \mathbb{Q})^* = \text{cl}(\mathbb{R} - \mathbb{Q}) = \mathbb{R} \not\subseteq \mathbb{R} - \mathbb{Q} = A \cup B$. Hence $A \cup B$ is not weakly $\mathcal{I}_g$ - ω -closed.

Theorem 4.9. In an ideal topological space $(X, \tau, \mathcal{I})$, a subset G is weakly $\mathcal{I}_g$ - ω -open if and only if $G \subseteq \text{int}^*(\text{cl}(G))$.

Proof. G is weakly $\mathcal{I}_g$ - ω -open $\Leftrightarrow X \setminus G$ is weakly $\mathcal{I}_g$ - ω -closed $\Leftrightarrow X \setminus G$ is $\text{pre}_{\mathcal{I}}^*$ -closed by (5) of Theorem 3.1 $\Leftrightarrow G$ is $\text{pre}_{\mathcal{I}}^*$ -open $\Leftrightarrow G \subseteq \text{int}^*(\text{cl}(G))$. $\square$

Theorem 4.10. In an ideal topological space $(X, \tau, \mathcal{I})$, if the subset G is weakly $\mathcal{I}_g$ - ω -closed, then $\text{cl}^*(\text{int}(G)) \setminus G$ is weakly $\mathcal{I}_g$ - ω -open in $(X, \tau, \mathcal{I})$.

Proof. Since G is weakly $\mathcal{I}_g$ - ω -closed, $\text{cl}^*(\text{int}(G)) \setminus G = \phi$ by (3) of Theorem 3.1. Thus $\text{cl}^*(\text{int}(G)) \setminus G$ is weakly $\mathcal{I}_g$ - ω -open in $(X, \tau, \mathcal{I})$. $\square$

Theorem 4.11. In an ideal topological space $(X, \tau, \mathcal{I})$, if G is weakly $\mathcal{I}_g$ - ω -open, then $\text{int}^*(\text{cl}(G)) \cup (X - G) = X$.

Proof. Since G is weakly $\mathcal{I}_g$ - ω -open, $G \subseteq \text{int}^*(\text{cl}(G))$ by Theorem 4.9. So $(X - G) \cup G \subseteq (X - G) \cup \text{int}^*(\text{cl}(G))$ which implies $X = (X - G) \cup \text{int}^*(\text{cl}(G))$. $\square$

Theorem 4.12. In an ideal topological space $(X, \tau, \mathcal{I})$, if G is weakly $\mathcal{I}_g$ - ω -open and H is a subset such that $\text{int}^*(\text{cl}(G)) \subseteq H \subseteq G$, then H is weakly $\mathcal{I}_g$ - ω -open.

Proof. Since G is weakly $\mathcal{I}_g$ - ω -open, $G \subseteq \text{int}^*(\text{cl}(G))$ by Theorem 4.9. By assumption $\text{int}^*(\text{cl}(G)) \subseteq H \subseteq G$. This implies $G \subseteq \text{int}^*(\text{cl}(G)) \subseteq H \subseteq G$. Thus $G = H$ and so H is weakly $\mathcal{I}_g$ - ω -open. $\square$

Corollary 4.13. In an ideal topological space $(X, \tau, \mathcal{I})$, if G is a weakly $\mathcal{I}_g$ - ω -open set and a closed set, then $\text{int}^*(G)$ is weakly $\mathcal{I}_g$ - ω -open.

Proof. Let G be a weakly $\mathcal{I}_g$ - ω -open set and a closed set in $(X, \tau, \mathcal{I})$. Then $\text{int}^*(\text{cl}(G)) = \text{int}^*(G) \subseteq \text{int}^*(G) \subseteq G$. Thus, by Theorem 4.12, $\text{int}^*(G)$ is weakly $\mathcal{I}_g$ - ω -open in $(X, \tau, \mathcal{I})$. $\square$

Definition 4.14. A subset H of an ideal topological space $(X, \tau, \mathcal{I})$ is called a $W_{\mathcal{I}}$ -set if $H = M \cup N$ where M is ω -closed and N is $\text{pre}_{\mathcal{I}}^*$ -open.

Proposition 4.15. Every $\text{pre}_{\mathcal{I}}^*$ -open (resp. ω -closed) set is a $W_{\mathcal{I}}$ -set.

Remark 4.16. The separate converses of Proposition 4.15 are not true in general as shown in the following example.

Example 4.17.

(1). Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 2.11 and $G = \mathbb{R} - \mathbb{Q}$. Since G is closed, it is ω -closed and hence a $W_{\mathcal{I}}$ -set. But $\text{int}^*(\text{cl}(G)) = \text{int}^*(G) = \mathbb{R} \setminus \text{cl}^*(\mathbb{Q}) = \mathbb{R} \setminus \text{cl}(\mathbb{Q}) = \mathbb{R} \setminus \mathbb{R} = \phi \not\subseteq G$. Hence $G = \mathbb{R} - \mathbb{Q}$ is not $\text{pre}_{\mathcal{I}}^*$ -open.

(2). In Example 4.5, for $G = \mathbb{R} - \mathbb{Q}$, $\text{int}^*(\text{cl}(G)) = \text{int}^*(\mathbb{R}) = \mathbb{R} \supseteq G$. Thus G is $\text{pre}_{\mathcal{I}}^*$ -open and hence a $W_{\mathcal{I}}$ -set. But G is not ω -closed for any $x \in \mathbb{Q}$ is a condensation point of G and $x \notin G$.

Remark 4.18. The following example shows that the concepts of $\text{pre}_{\mathcal{I}}^*$ -openness and ω -closedness are independent of each other.

Example 4.19. In Example 4.17(1), $G = \mathbb{R} - \mathbb{Q}$ is ω -closed but not $\text{pre}_{\mathcal{I}}^*$ -open.

In Example 4.17(2), $G = \mathbb{R} - \mathbb{Q}$ is $\text{pre}_{\mathcal{I}}^*$ -open but not ω -closed.

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