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Corrections on Decompositions of ω-continuity*

Research Article

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Abstract: In 2009, Noiri et al [3] introduced some weaker forms of ω-open sets in topological spaces. In this paper, we introduce some new subsets of τ_ω in topological spaces. Using the weaker forms of ω-open sets and the new subsets of τ_ω, we obtain some new decompositions of ω-continuity.

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1. Introduction

Hdeib [2] introduced the concepts of ω-closed and ω-open sets in topological spaces. Noiri et al [3] introduced the concepts of α-ω-open, pre-ω-open, β-ω-open and b-ω-open sets in topological spaces and investigated their properties. Moreover, they used them to obtain decompositions of continuity. Quite Recently, Ravi et al [5] introduced another weaker form of ω-open sets called semi-ω-open sets and proved that the class of semi-ω-open sets is stronger form of the class of b-ω-open sets. Also, they studied their topological properties. Ravi et al [4] introduced some subsets of τ_ω and studied their properties. Further more they used them to obtain some decompositions of continuity. In this paper, we introduce some new subsets of τ_ω in topological spaces. Using the weaker forms of ω-open sets and the new subsets of τ_ω, we obtain some new decompositions of ω-continuity.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Q}^\star$, $\mathbb{Q}^\star_+$) denotes the set of all real numbers (resp. the set of all natural numbers, the set of all rational numbers, the set of all irrational numbers, the set of all positive irrational numbers). By a space (X, τ), we always mean a topological space (X, τ) with no separation properties assumed. If H $\subset$ X, cl(H) and int(H) will, respectively, denote the closure and interior of H in (X, τ). τ_u denotes the usual topology on $\mathbb{R}$.

* *This paper is published in the South Asian Journal of Mathematics, volume 6, issue 5, year 2016, pages 215-228. Examples 4.32(2) and 5.10 of this paper were found to be incorrect in the published paper. So the incorrections are rightly corrected out in this paper for readers.*
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Definition 2.1 ([6]). *Let H be a subset of a space (X, τ), a point p in X is called a condensation point of H if for each open set U containing p, $U \cap H$ is uncountable.*

Definition 2.2 ([2]). *A subset H of a space (X, τ) is called ω-closed if it contains all its condensation points. The complement of an ω-closed set is called ω-open.*

It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.

Definition 2.3 ([3]). *A subset H of a space (X, τ) is called*

(1). α-ω-open if $H \subset int_\omega(cl(int_\omega(H)))$;

(2). pre-ω-open if $H \subset int_\omega(cl(H))$;

(3). β-ω-open if $H \subset cl(int_\omega(cl(H)))$;

(4). b-ω-open if $H \subset int_\omega(cl(H)) \cup cl(int_\omega(H))$.

Definition 2.4 ([5]). *A subset H of a space (X, τ) is called semi-ω-open if $H \subset cl(int_\omega(H))$.*

Definition 2.5 ([4]). *A subset H of a space (X, τ) is called*

(1). an $\omega^\#$-t-set if $int(H) = cl(int_\omega(H))$;

(2). an $\omega^\#$-$\mathcal{B}$-set if $H = U \cap V$, where $U \in \tau$ and V is an $\omega^\#$-t-set.

Definition 2.6 ([1]). *A subset H of a space (X, τ) is called locally closed if $H = U \cap V$, where U is open and V is closed.*

Definition 2.7 ([5]). *A subset H of a space (X, τ) is called an $\omega^\star$-t-set if $int_\omega(cl(H)) = int_\omega(H)$.*

Definition 2.8 ([3]). *A subset H of a space (X, τ) is called an ω-t-set if $int(H) = int_\omega(cl(H))$.*

Definition 2.9 ([5]). *A subset H of a space (X, τ) is called semi-ω-regular if H is semi-ω-open and an $\omega^\star$-t-set.*

Remark 2.10 ([3, 5]). *The diagram holds for subsets of a space (X, τ):*

$$\begin{array}{ccccccc}
\textbf{open} & \longrightarrow & \alpha\text{-}\omega\textbf{-open} & \longrightarrow & \textbf{pre-}\omega\textbf{-open} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\omega\textbf{-open} & \longrightarrow & \textbf{semi-}\omega\textbf{-open} & \longrightarrow & \textbf{b-}\omega\textbf{-open} & \longrightarrow & \beta\text{-}\omega\textbf{-open}
\end{array}$$

In this diagram, none of the implications is reversible.

Theorem 2.11 ([5]). *Let H be a subset of a space (X, τ). Then H is α-ω-open if and only if it is semi-ω-open and pre-ω-open.*

Remark 2.12 ([4]). *(1). In $\mathbb{R}$ with usual topology τ_u, a subset H with $int(H) = \phi$ is an $\omega^\#$-t-set if and only if $int(H) = \phi = int_\omega(H)$.*

(2). In $\mathbb{R}$ with usual topology τ_u, there is no proper subset H, with $int(H) \neq \phi$ which is an $\omega^\#$-t-set. (or) The only subset in $\mathbb{R}$, with nonempty interior, which is an $\omega^\#$-t-set is $\mathbb{R}$ itself.

Example 2.13 ([4]). *In $\mathbb{R}$ with usual topology τ_u, $H = \mathbb{Q}^\star$ is not an $\omega^\#$-t-set by (1) of Remark 2.12 since $int(H) = \phi \neq int_\omega(H)$.*

3. Generalizations of ω-open Sets

Definition 3.1. *A subset H of a space (X, τ) is called an ω-$\mathcal{B}^{\star\star}$-set if $H = U \cap V$, where $U \in \tau_\omega$ and V is an $\omega^{\#}$-t-set.*

Remark 3.2. *In a space (X, τ),*

(1). Every ω-open set is an ω-$\mathcal{B}^{\star\star}$-set.

(2). Every $\omega^{\#}$-t-set is an ω-$\mathcal{B}^{\star\star}$-set.

Example 3.3. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = [0, 1] \cap \mathbb{Q}$ is $\omega^{\#}$-t-set by (1) of Remark 2.12 and hence an ω-$\mathcal{B}^{\star\star}$-set by (2) of Remark 3.2.

(2). $H = [0, 1]$ is not an ω-$\mathcal{B}^{\star\star}$-set. If $H = U \cap V$ where $U \in \tau_\omega$ and V is an $\omega^{\#}$-t-set, then $H \subset V$. Since $int(H) \neq \phi$, $int(V) \neq \phi$. Hence $V = \mathbb{R}$ by (2) of Remark 2.12. Thus $H = U \cap \mathbb{R} = U$ where $U \in \tau_\omega$. Hence $H \in \tau_\omega$ which is a contradiction. So $H = [0, 1]$ is not an ω-$\mathcal{B}^{\star\star}$-set.

Remark 3.4. *The converses of (1) and (2) in Remark 3.2 are not true as seen from the following Example.*

Example 3.5. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = [0, 1] \cap \mathbb{Q}$ is an ω-$\mathcal{B}^{\star\star}$-set by (1) of Example 3.3. But H is not ω-open since $H \neq int_\omega(H)$.

(2). $H = (0, 1)$ is ω-open and hence an ω-$\mathcal{B}^{\star\star}$-set by (1) of Remark 3.2. But H is not an $\omega^{\#}$-t-set.

Proposition 3.6. *For a subset H of a space (X, τ), the following are equivalent:*

(1). H is ω-open;

(2). H is semi-ω-open and an ω-$\mathcal{B}^{\star\star}$-set.

Proof.
(1)⇒(2): (2) follows by Remark 2.10 and (1) of Remark 3.2.
(2)⇒(1): Given H is an ω-$\mathcal{B}^{\star\star}$-set. So $H = U \cap V$ where $U \in \tau_\omega$ and V is an $\omega^{\#}$-t-set. Then $H \subset U = int_\omega(U)$. Also H is semi-$\omega$-open implies $H \subset cl(int_\omega(H)) \subset cl(int_\omega(V)) = int(V)$ by assumption. Thus $H \subset int_\omega(U) \cap int(V) \subset int_\omega(U) \cap int_\omega(V) = int_\omega(U \cap V) = int_\omega(H)$ and hence H is ω-open. □

Remark 3.7. *The following Example shows that the concepts of semi-ω-openness and being an ω-$\mathcal{B}^{\star\star}$-set are independent.*

Example 3.8. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = [0, 1] \cap \mathbb{Q}$ is an ω-$\mathcal{B}^{\star\star}$-set by (1) of Remark 3.5. But H is not semi-ω-open since $H \nsubseteq cl(int_\omega(H)) = cl(\phi) = \phi$.

(2). For $H = [0, 1]$, $cl(int_\omega(H)) = cl((0, 1)) = [0, 1]$. Thus $H \subset cl(int_\omega(H))$ and H is semi-ω-open. But H is not an ω-$\mathcal{B}^{\star\star}$-set by (2) of Example 3.3.

Definition 3.9. *A subset H of a space (X, τ) is called*

(1). an $\omega^{\star\star}$-t-set if $int_\omega(H) = cl(int_\omega(H))$.

(2). an $\omega^{\star\star}$-$\mathcal{B}$-set if $H = U \cap V$, where $U \in \tau_\omega$ and V is an $\omega^{\star\star}$-t-set.

Example 3.10. *In $\mathbb{R}$ with usual topology τ_u,* *(1).* $H = [0,1] \cap \mathbb{Q}$ *is an* $\omega^{\star\star}$*-t-set since* $int_\omega(H) = cl(int_\omega(H)) = \phi$.

(2). $H = [0,1]$ *is not an* $\omega^{\star\star}$*-t-set since* $int_\omega(H) = (0,1)$ *and* $cl(int_\omega(H)) = [0,1]$.

Remark 3.11. *In a space* (X, τ),

(1). Every ω*-open set is an* $\omega^{\star\star}$*-*$\mathcal{B}$*-set.*

(2). Every $\omega^{\star\star}$*-t-set is an* $\omega^{\star\star}$*-*$\mathcal{B}$*-set.*

Example 3.12. *(1). In* $\mathbb{R}$ *with usual topology* τ_u, $H = [0,1] \cap \mathbb{Q}$ *is an* $\omega^{\star\star}$*-t-set by (1) of Example 3.10 and hence an* $\omega^{\star\star}$*-*$\mathcal{B}$*-set by (2) of Remark 3.11.*

(2). In $\mathbb{R}$ *with the topology* $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$, $H = \mathbb{Q} \cup \{\sqrt{2}\}$ *is not an* $\omega^{\star\star}$*-*$\mathcal{B}$*-set.*

If $H = U \cap V$ *where* $U \in \tau_\omega$ *and* V *is an* $\omega^{\star\star}$*-t-set then* $H \subset V$ *and* $cl(int_\omega(H)) \subset cl(int_\omega(V)) = int_\omega(V)$*. Hence* $cl(\mathbb{Q}) \subset int_\omega(V)$ *and we have* $\mathbb{R} \subset int_\omega(V)$*. Thus* $\mathbb{R} = V$ *and* $H = U \cap \mathbb{R} = U \in \tau_\omega$ *which is a contradiction since* H *is not* ω*-open. This proves that* H *is not an* $\omega^{\star\star}$*-*$\mathcal{B}$*-set.*

Remark 3.13. *The converses of (1) and (2) in Remark 3.11 are not true as seen from the following Example.*

Example 3.14. *In* $\mathbb{R}$ *with usual topology* τ_u,

(1). $H = [0,1] \cap \mathbb{Q}$ *is an* $\omega^{\star\star}$*-*$\mathcal{B}$*-set by Example 3.12(1). But* H *is not* ω*-open since* $H \neq int_\omega(H)$.

(2). $H = (0,1)$ *is* ω*-open and hence an* $\omega^{\star\star}$*-*$\mathcal{B}$*-set by (1) of Remark 3.11. But* H *is not an* $\omega^{\star\star}$*-t-set.*

Proposition 3.15. *For a subset* H *of a space* (X, τ)*, the following are equivalent:*

(1). H *is* ω*-open;*

(2). H *is semi-*ω*-open and an* $\omega^{\star\star}$*-*$\mathcal{B}$*-set.*

Proof.
(1)⇒(2): (2) follows by Remark 2.10 and (1) of Remark 3.11.
(2)⇒(1): Given H is an $\omega^{\star\star}$-$\mathcal{B}$-set. So $H = U \cap V$ where $U \in \tau_\omega$ and $int_\omega(V) = cl(int_\omega(V))$. Then $H \subset U = int_\omega(U)$. Also H is semi-$\omega$-open implies $H \subset cl(int_\omega(H)) \subset cl(int_\omega(V)) = int_\omega(V)$ by assumption. Thus $H \subset int_\omega(U) \cap int_\omega(V) = int_\omega(U \cap V) = int_\omega(H)$ and hence H is ω-open. □

Remark 3.16. *The following Example shows that the concepts of semi-*ω*-openness and being an* $\omega^{\star\star}$*-*$\mathcal{B}$*-set are independent.*

Example 3.17.

(1). In $\mathbb{R}$ *with usual topology* τ_u, $H = [0,1] \cap \mathbb{Q}$ *is* $\omega^{\star\star}$*-*$\mathcal{B}$*-set by (1) of Example 3.14. But* H *is not semi-*ω*-open since* $H \nsubseteq cl(int_\omega(H)) = \phi$.

(2). In $\mathbb{R}$ *with the topology* $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$*, for* $H = \mathbb{Q} \cup \{\sqrt{2}\}$, $cl(int_\omega(H)) = cl(\mathbb{Q}) = \mathbb{R}$ *and* $H \subset cl(int_\omega(H))$*. Hence* H *is semi-*ω*-open. But* H *is not an* $\omega^{\star\star}$*-*$\mathcal{B}$*-set by (2) of Example 3.12.*

Proposition 3.18. *In a space* (X, τ)*, every* $\omega^{\#}$*-t-set is an* $\omega^{\star\star}$*-t-set.*

Example 3.19. *In* $\mathbb{R}$ *with the topology* $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$*, for* $H = \mathbb{Q}^\star$, $cl(int_\omega(H)) = cl(H) = H = int_\omega(H)$*. Hence* H *is an* $\omega^{\star\star}$*-t-set. But* $int(H) = \phi \neq cl(int_\omega(H))$*. Thus* H *is not an* $\omega^{\#}$*-t-set.*

Proposition 3.20. *In a space (X, τ), every ω-$\mathcal{B}^{\star\star}$-set is an $\omega^{\star\star}$-$\mathcal{B}$-set.*

Proof. It follows from Proposition 3.18. □

Example 3.21. *Let $X = A \cup B$ where $A = (0, 1)$ and $B = (1, 2)$ and $\tau = \{\phi, X, A, A \cap \mathbb{Q}, (A \cap \mathbb{Q}) \cup B\}$. Then for $H = (A \cap \mathbb{Q}^\star) \cup (B \cap \mathbb{Q})$, $int_\omega(H) = A \cap \mathbb{Q}^\star$ and $cl(int_\omega(H)) = cl(A \cap \mathbb{Q}^\star) = A \cap \mathbb{Q}^\star = int_\omega(H)$. Thus H is an $\omega^{\star\star}$-t-set and hence H is an $\omega^{\star\star}$-$\mathcal{B}$-set by (2) of Remark 3.11. We prove that H is not an ω-$\mathcal{B}^{\star\star}$-set. If $H = U \cap V$ where $U \in \tau_\omega$ and V is an $\omega^{\#}$-t-set, then $H \subset V$. This implies $cl(int_\omega(H)) \subset cl(int_\omega(V)) = int(V)$ by assumption. Thus $int_\omega(H) \subset int(V)$ and $int(V)$ is an open set containing $int_\omega(H) = A \cap \mathbb{Q}^\star$. So $int(V) = (0, 1)$ or X. If $int(V) = (0, 1)$ then $int(V) = cl(int_\omega(A))$ is a closed set which is a contradiction since $int(V) = (0, 1)$ is not closed. Hence $int(V) = X$ and $V = X$. Thus $H = U \cap X = U \in \tau_\omega$ which is a contradiction since H is not ω-open. This proves that H is not an ω-$\mathcal{B}^{\star\star}$-set.*

4. New Subsets of τ_ω

Definition 4.1. *A subset H of a space (X, τ) is called an $\omega^\star$-$\mathcal{B}$-set if $H = U \cap V$, where $U \in \tau_\omega$ and V is an $\omega^\star$-t-set.*

Remark 4.2. *In a space (X, τ),*

(1). Every ω-open set is an $\omega^\star$-$\mathcal{B}$-set.

(2). Every $\omega^\star$-t-set is an $\omega^\star$-$\mathcal{B}$-set.

Example 4.3. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = (0, 1]$ is an $\omega^\star$-t-set and hence an $\omega^\star$-$\mathcal{B}$-set by (2) of Remark 4.2.

(2). $H = \mathbb{Q}$ is not an $\omega^\star$-$\mathcal{B}$-set. If $H = U \cap V$ where $U \in \tau_\omega$ and V is an $\omega^\star$-t-set, then $H \subset V$ and $int_\omega(cl(H)) \subset int_\omega(cl(V))$. Hence $\mathbb{R} \subset int_\omega(cl(V)) = int_\omega(V)$ and thus $\mathbb{R} = V$ and $H = U \cap \mathbb{R} = U \in \tau_\omega$ which is a contradiction since H is not ω-open. This proves that H is not an $\omega^\star$-$\mathcal{B}$-set.

Remark 4.4. *The converses of (1) and (2) in Remark 4.2 are not true as seen from the following Example.*

Example 4.5. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = (0, 1]$ is an $\omega^\star$-$\mathcal{B}$-set by (1) of Example 4.3. But H is not ω-open.

(2). $H = \mathbb{Q}^\star$ is ω-open and hence an $\omega^\star$-$\mathcal{B}$-set by (1) of Remark 4.2. But H is not an $\omega^\star$-t-set.

Proposition 4.6. *For a subset H of a space (X, τ), the following are equivalent:*

(1). H is ω-open;

(2). H is pre-ω-open and an $\omega^\star$-$\mathcal{B}$-set.

Proof.
(1)⇒(2): (2) follows by Remark 2.10 and (1) of Remark 4.2.
(2)⇒(3): Given H is an $\omega^\star$-$\mathcal{B}$-set. So $H = U \cap V$ where $U \in \tau_\omega$ and $int_\omega(cl(V)) = int_\omega(V)$. Then $H \subset U = int_\omega(U)$. Also H is pre-$\omega$-open implies $H \subset int_\omega(cl(H)) \subset int_\omega(cl(V)) = int_\omega(V)$ by assumption. Thus $H \subset int_\omega(U) \cap int_\omega(V) = int_\omega(U \cap V) = int_\omega(H)$ and hence H is ω-open. □

Remark 4.7. *The following Example shows that the concepts of pre-ω-openness and being an $\omega^\star$-$\mathcal{B}$-set are independent.*

Example 4.8. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = \mathbb{Q}$ is pre-ω-open but not an $\omega^\star$-$\mathcal{B}$-set by (2) of Example 4.3.

(2). $H = (0, 1]$ is an $\omega^\star$-$\mathcal{B}$-set by (1) of Example 4.3. But H is not pre-ω-open since $H \nsubseteq int_\omega(cl(H)) = (0, 1)$.

Definition 4.9. *A subset H of a space (X, τ) is called an ω-$\mathcal{B}^\star$-set if $H = U \cap V$, where $U \in \tau_\omega$ and V is ω-t-set.*

Remark 4.10. *In a space (X, τ),*

(1). Every ω-open set is an ω-$\mathcal{B}^\star$-set.

(2). Every ω-t-set is an ω-$\mathcal{B}^\star$-set.

Example 4.11. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = (0, 1]$ is an ω-t-set and hence an ω-$\mathcal{B}^\star$-set by (2) of Remark 4.10 .

(2). $H = \mathbb{Q}$ is not an ω-$\mathcal{B}^\star$-set. If $H = U \cap V$ where $U \in \tau_\omega$ and V is an ω-t-set then $H \subset V$ and $int_\omega(cl(H)) \subset int_\omega(cl(V))$. Hence $\mathbb{R} \subset int_\omega(cl(V)) = int(V)$. Thus $\mathbb{R} = V$ and $H = U \cap \mathbb{R} = U \in \tau_\omega$ which is a contradiction since $\mathbb{Q}$ is not ω-open. This proves that $H = \mathbb{Q}$ is not an ω-$\mathcal{B}^\star$-set.

Remark 4.12. *The converses of (1) and (2) in Remark 4.10 are not true as seen from the following Example.*

Example 4.13. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = (0, 1]$ is an ω-$\mathcal{B}^\star$-set by (1) of Example 4.11. But H is not ω-open.

(2). $H = \mathbb{Q}^\star$ is ω-open and hence an ω-$\mathcal{B}^\star$-set by (1) of Remark 4.10. But H is not an ω-t-set.

Proposition 4.14. *For a subset H of a space (X, τ), the following are equivalent:*

(1). H is ω-open;

(2). H is pre-ω-open and an ω-$\mathcal{B}^\star$-set.

Proof.
(1)⇒(2): (2) follows by Remark 2.10 and (1) of Remark 4.10.
(2)⇒(1): Given H is an ω-$\mathcal{B}^\star$-set. So $H = U \cap V$ where $U \in \tau_\omega$ and $int_\omega(cl(V)) = int(V)$. Then $H \subset U = int_\omega(U)$. Also H is pre-$\omega$-open implies $H \subset int_\omega(cl(H)) \subset int_\omega(cl(V)) = int(V)$ by assumption. Thus $H \subset int_\omega(U) \cap int(V) \subset int_\omega(U) \cap int_\omega(V) = int_\omega(U \cap V) = int_\omega(H)$ and hence H is ω-open. □

Remark 4.15. *The following Example shows that the concepts of pre-ω-openness and being an ω-$\mathcal{B}^\star$-set are independent.*

Example 4.16. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = (0, 1]$ is an ω-$\mathcal{B}^\star$-set by (1) of Example 4.11. But H is not pre-ω-open by (2) of Example 4.8.

(2). $H = \mathbb{Q}$ is pre-ω-open by (1) of Example 4.8. But H is not an ω-$\mathcal{B}^\star$-set by (2) of Example 4.11.

Definition 4.17. *A subset H of a space (X, τ) is called*

(1). ω-$\mathcal{R}$-closed [5] if $H = cl(int_\omega(H))$.

(2). ω-$\mathcal{R}$-open if $H = int(cl_\omega(H))$.

The complement of an ω-$\mathcal{R}$-open set is called ω-$\mathcal{R}$-closed.

Example 4.18. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = [0, 1]$ is ω-$\mathcal{R}$-closed.

(2). $H = (0, 1]$ is not ω-$\mathcal{R}$-closed.

Definition 4.19. *A subset H of a space (X, τ) is called a $\mathcal{H}^{\star}_{\omega}$-set if $H = U \cap V$, where $U \in \tau_\omega$ and V is ω-$\mathcal{R}$-closed.*

Remark 4.20. *In a space (X, τ),*

(1). Every ω-open set is a $\mathcal{H}^{\star}_{\omega}$-set.

(2). Every ω-$\mathcal{R}$-closed set is a $\mathcal{H}^{\star}_{\omega}$-set.

(3). Every ω-$\mathcal{R}$-closed set is closed by definition.

(4). A nonempty subset H is ω-$\mathcal{R}$-closed if and only if $int_\omega(H) \neq \phi$.

Example 4.21. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = [0, 1]$ is ω-$\mathcal{R}$-closed by (1) of Example 4.18 and hence a $\mathcal{H}^{\star}_{\omega}$-set by (2) of Remark 4.20.

(2). $H = \mathbb{Q}$ is not a $\mathcal{H}^{\star}_{\omega}$-set. If $H = U \cap V$ where $U \in \tau_\omega$ and V is ω-$\mathcal{R}$-closed, then $H \subset V$. Hence $cl(H) \subset cl(V) = V$ by (3) of Remark 4.20. Thus $\mathbb{R} \subset V$ and so $\mathbb{R} = V$. Then we have $H = U \cap \mathbb{R} = U \in \tau_\omega$ which is a contradiction since $H = \mathbb{Q}$ is not ω-open. This proves that $H = \mathbb{Q}$ is not a $\mathcal{H}^{\star}_{\omega}$-set.

Remark 4.22. *The converses of (1) and (2) in Remark 4.20 are not true as seen from the following Example.*

Example 4.23. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = [0, 1]$ is $\mathcal{H}^{\star}_{\omega}$-set by (1) of Example 4.21. But H is not ω-open.

(2). $H = (0, 1)$ is ω-open and hence $\mathcal{H}^{\star}_{\omega}$-set by (1) of Remark 4.20. But H is not ω-$\mathcal{R}$-closed.

Theorem 4.24. *For a subset H of space (X, τ), the following are equivalent:*

(1). H is ω-open;

(2). H is α-ω-open and a $\mathcal{H}^{\star}_{\omega}$-set.

(3). H is pre-ω-open and a $\mathcal{H}^{\star}_{\omega}$-set.

Proof.
(1)⇒(2): (2) follows by Remark 2.10 and (1) of Remark 4.20.
(2)⇒(3): (3) follows by Remark 2.10.
(3)⇒(1): Given H is a $\mathcal{H}^{\star}_{\omega}$-set. So $H = U \cap V$ where $U \in \tau_\omega$ and $V = cl(int_\omega(V))$. Then $H \subset U = int_\omega(U)$. Also H is pre-$\omega$-open implies $H \subset int_\omega(cl(H)) \subset int_\omega(cl(V)) = int_\omega(cl(cl(int_\omega(V))))$ (by assumption) $= int_\omega(cl(int_\omega(V))) = int_\omega(V)$. Thus $H \subset int_\omega(U) \cap int_\omega(V) = int_\omega(U \cap V) = int_\omega(H)$ and hence H is ω-open. □

Remark 4.25. *The following Example shows that*

(1). the concepts of α-ω-openness and being a $\mathcal{H}^{\star}_{\omega}$-set are independent.

(2). the concepts of pre-ω-openness and being a $\mathcal{H}^\star_\omega$-set are independent.

Example 4.26. *(1). In $\mathbb{R}$ with usual topology τ_u, $H = [0, 1]$ is a $\mathcal{H}^\star_\omega$-set by (1) of Example 4.23. But H is not α-ω-open.*

(2). In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}\}$, for $H = \mathbb{Q}$, $int_\omega(cl(int_\omega(H))) = int_\omega(cl(\mathbb{N})) = int_\omega(\mathbb{R}) = \mathbb{R}$ and so $H \subset int_\omega(cl(int_\omega(H)))$. Hence H is α-ω-open. But H is not a $\mathcal{H}^\star_\omega$-set. If $\mathbb{Q} = H = U \cap V$ where $U \in \tau_\omega$ and V is ω-$\mathcal{R}$-closed, then we have $H \subset V$ and $cl(H) \subset cl(V) = V$ by (3) of Remark 4.20. Thus $\mathbb{R} \subset V$ and so $\mathbb{R} = V$. Then we have $H = U \cap \mathbb{R} = U \in \tau_\omega$ which is a contradiction since H is not ω-open. This proves that H is not a $\mathcal{H}^\star_\omega$-set.

Example 4.27.

(1). In $\mathbb{R}$ with usual topology τ_u, $H = [0, 1]$ is a $\mathcal{H}^\star_\omega$-set by (1) of Example 4.26. But H is not pre-ω-open.

(2). In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}\}$, $H = \mathbb{Q}$ is pre-ω-open but not a $\mathcal{H}^\star_\omega$-set by (2) of Example 4.26.

Definition 4.28. *A subset H of a space (X, τ) is called an ω-$\mathcal{AB}^\#$-set if $H = U \cap V$, where $U \in \tau_\omega$ and V is semi-ω-regular.*

Remark 4.29. *In a space (X, τ),*

(1). Every ω-open set is an ω-$\mathcal{AB}^\#$-set.

(2). Every semi-ω-regular set is an ω-$\mathcal{AB}^\#$-set.

Example 4.30. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = (0, 1]$ is both semi-ω-open and an $\omega^\star$-t-set. So H is semi-ω-regular and hence an ω-$\mathcal{AB}^\#$-set by (2) of Remark 4.29.

(2). $H = \mathbb{Q}$ is not an ω-$\mathcal{AB}^\#$-set. If $H = U \cap V$ where $U \in \tau_\omega$ and V is semi-ω-regular, then $H \subset V$ and $int_\omega(cl(H)) \subset int_\omega(cl(V)) = int_\omega(V)$ by assumption. Hence $\mathbb{R} \subset int_\omega(V) \subset V$ and $\mathbb{R} = V$. Thus $H = U \cap \mathbb{R} = U \in \tau_\omega$ which is a contradiction since H is not ω-open. This proves that $H = \mathbb{Q}$ is not an ω-$\mathcal{AB}^\#$-set.

Remark 4.31. *The converses of (1) and (2) in Remark 4.29 are not true as seen from the following Example.*

Example 4.32.

(1). In $\mathbb{R}$ with usual topology τ_u, $H = (0, 1]$ is ω-$\mathcal{AB}^\#$-set by (1) of Example 4.30. But H is not ω-open.

(2). In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$, $H = \mathbb{Q}$ is ω-open and hence an ω-$\mathcal{AB}^\#$-set by (1) of Remark 4.29. But $int_\omega(H) = H$ and $int_\omega(cl(H)) = int_\omega(\mathbb{R}) = \mathbb{R}$ and $int_\omega(H) \neq int_\omega(cl(H))$. Thus H is not an $\omega^\star$-t-set and hence not semi-ω-regular.

Theorem 4.33. *For a subset H of a space (X, τ), the following are equivalent:*

(1). H is ω-open;

(2). H is α-ω-open and an ω-$\mathcal{AB}^\#$-set.

(3). H is pre-ω-open and an ω-$\mathcal{AB}^\#$-set.

Proof.
(1)⇒(2): (2) follows by Remark 2.10 and (1) of Remark 4.29.
(2)⇒(3): (3) follows by Remark 2.10.
(3)⇒(1): Given H is an ω-$\mathcal{AB}^\#$-set. So $H = U \cap V$ where $U \in \tau_\omega$ and V is semi-ω-regular. Then $H \subset U = int_\omega(U)$. Also H is pre-$\omega$-open implies $H \subset int_\omega(cl(H)) \subset int_\omega(cl(V)) = int_\omega(V)$ by assumption. Thus $H \subset int_\omega(U) \cap int_\omega(V) = int_\omega(U \cap V) = int_\omega(H)$ and hence H is ω-open. □

Remark 4.34. *The following Example shows that*

(1). the concepts of α-ω-openness and being an ω-$\mathcal{AB}^{\#}$-set are independent.

(2). the concepts of pre-ω-openness and being an ω-$\mathcal{AB}^{\#}$-set are independent.

Example 4.35.

(1). In $\mathbb{R}$ with usual topology τ_u, $H = (0, 1]$ is an ω-$\mathcal{AB}^{\#}$-set by (1) of Example 4.32. But H is not α-ω-open.

(2). In $\mathbb{R}$ with topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}\}$, for $H = \mathbb{Q}$, $int_\omega(cl(int_\omega(H))) = int_\omega(cl(\mathbb{N})) = int_\omega(\mathbb{R}) = \mathbb{R} \supset H$. Thus H is α-ω-open. But H is not an ω-$\mathcal{AB}^{\#}$-set. If $H = U \cap V$ where $U \in \tau_\omega$ and V is semi-ω-regular then $H \subset V$ and $int_\omega(cl(H)) \subset int_\omega(cl(V)) = int_\omega(V)$ by assumption. Hence $\mathbb{R} \subset int_\omega(V) \subset V$ and $V = \mathbb{R}$. Thus $H = U \cap \mathbb{R} = U \in \tau_\omega$ which is a contradiction since H is not ω-open. This proves that $H = \mathbb{Q}$ is not an ω-$\mathcal{AB}^{\#}$-set.

(3). In $\mathbb{R}$ with usual topology τ_u, $H = \mathbb{Q}$ is pre-ω-open but not an ω-$\mathcal{AB}^{\#}$-set by (2) of Example 4.30.

(4). In $\mathbb{R}$ with usual topology τ_u, $H = (0, 1]$ is an ω-$\mathcal{AB}^{\#}$-set but not pre-ω-open.

5. ω-extremally Disconnected Space

Definition 5.1. *A subset H of a space (X, τ) is called locally ω-closed if $H = U \cap V$, where $U \in \tau_\omega$ and V is closed.*

Remark 5.2. *In a space (X, τ),*

(1). Every ω-open set is locally ω-closed.

(2). Every closed set is locally ω-closed.

Example 5.3.

(1). In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$, $H = \mathbb{Q}$ is ω-open and hence locally ω-closed by (1) of Remark 5.2.

(2). In $\mathbb{R}$ with usual topology τ_u, $H = \mathbb{Q}$ is not locally ω-closed. If $H = U \cap V$ where $U \in \tau_\omega$ and V is closed, then we have $H \subset V$ and $cl(H) \subset cl(V) = V$ by assumption. Thus $\mathbb{R} \subset V$ and so $\mathbb{R} = V$. But $H = U \cap \mathbb{R} = U \in \tau_\omega$ which is a contradiction since $H = \mathbb{Q}$ is not ω-open. This proves that $H = \mathbb{Q}$ is not locally ω-closed.

Remark 5.4. *The converses of (1) and (2) in Remark 5.2 are not true as seen from the following Example.*

Example 5.5. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = [0, 1]$ is closed and hence locally ω-closed by (2) of Remark 5.2. But H is not ω-open.

(2). $H = \mathbb{Q}^\star$ is ω-open and hence locally ω-closed by (1) of Remark 5.2. But H is not closed since $H \neq cl(H) = \mathbb{R}$.

Proposition 5.6. *For a subset H of a space (X, τ), the following are equivalent:*

(1). H is ω-open;

(2). H is pre-ω-open and locally ω-closed.

Proof.

(1)⇒(2): (2) follows by Remark 2.10 and (1) of Remark 5.2.

(2)⇒(1): Given H is locally ω-closed. So $H = U \cap V$ where $U \in \tau_\omega$ and $cl(V) = V$. Then $H \subset U = int_\omega(U)$. Also H is pre-$\omega$-open implies $H \subset int_\omega(cl(H)) \subset int_\omega(cl(V)) = int_\omega(V)$ by assumption. Thus $H \subset int_\omega(U) \cap int_\omega(V) = int_\omega(U \cap V) = int_\omega(H)$ and hence H is ω-open. □

Remark 5.7. *The following Example shows that the concepts of pre-ω-openness and locally ω-closedness are independent.*

Example 5.8. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = [0, 1]$ is closed and hence locally ω-closed by (2) of Remark 5.2. But H is not pre-ω-open.

(2). $H = \mathbb{Q}$ is pre-ω-open but not locally ω-closed, by (2) of Example 5.3.

Proposition 5.9. *Every locally closed set is locally ω-closed.*

Proof. It follows from the fact that every open set is ω-open. □

The converse of Proposition 5.9 is not true follows from the following Example.

Example 5.10. *In $\mathbb{R}$ with usual topology τ_u, $H = \mathbb{Q}^\star$ is locally ω-closed by (2) of Example 5.5. If $H = U \cap V$ where U is open and V is closed, then $H \subseteq V$ and $cl(H) \subseteq cl(V) = V$ by assumption on V. Thus $\mathbb{R} \subseteq V$ and so $V = \mathbb{R}$. Then $H = U \cap \mathbb{R} = U$ which implies that H is open. This is a contradiction since $H = \mathbb{Q}^\star$ is not open. Thus $H = \mathbb{Q}^\star$ is not locally closed.*

Definition 5.11. *A subset H of a space (X, τ) is called strong β-ω-open if $H \subset cl(int_\omega(cl_\omega(H)))$.*

Example 5.12. *In $\mathbb{R}$ with usual topology τ_u,*

(1). $H = (0, 1]$ is strong β-ω-open.

(2). $H = \mathbb{Q}$ is not strong β-ω-open.

Proposition 5.13. *In a space (X, τ), every strong β-ω-open set is β-ω-open.*

Proof. Let H be a strong β-ω-open set. Then $H \subset cl(int_\omega(cl_\omega(H))) \subset cl(int_\omega(cl(H)))$. Thus H is β-ω-open. □

Example 5.14. *In $\mathbb{R}$ with usual topology τ_u, $H = \mathbb{Q}$ is β-ω-open set but not strong β-ω-open.*

Proposition 5.15. *In a space (X, τ), every semi-ω-open set is strong β-ω-open.*

Proof. Let H be a semi-ω-open set. Then $H \subset cl(int_\omega(H)) \subset cl(int_\omega(cl_\omega(H)))$. Thus H is a strong β-ω-open. □

Example 5.16. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}^\star_+$ is strong β-ω-open but not semi-ω-open.*
For, $cl_\omega(H) = \mathbb{R}$ and $cl(int_\omega(cl_\omega(H))) = cl(int_\omega(\mathbb{R})) = cl(\mathbb{R}) = \mathbb{R}$. Thus $H \subset cl(int_\omega(cl_\omega(H)))$ and hence H is strong β-ω-open. But $int_\omega(H) = \phi$ and $cl(int_\omega(H)) = cl(\phi) = \phi$. Thus $H \not\subseteq cl(int_\omega(H))$ and hence H is not semi ω-open.

Definition 5.17. *A space (X, τ) is called ω-extremally disconnected if the closure of every ω-open subset H of X is ω-open.*

Theorem 5.18. *For a space (X, τ), the following are equivalent:*

(1). X is ω-extremally disconnected.

(2). int(H) is ω-closed for every ω-closed subset H of X.

(3). $cl(int_\omega(H)) \subset int_\omega(cl(H))$ for every subset H of X.

(4). Every semi-ω-open set is pre-ω-open.

(5). The closure of every strong β-ω-open subset of X is ω-open.

(6). Every strong β-ω-open set is pre-ω-open.

(7). For every subset H of X, H is α-ω-open if and only if it is semi-ω-open.

Proof.
(1)⇒(2): Let $H \subset X$ be a ω-closed. Then $X \backslash H$ is ω-open. By (1), $cl(X \backslash H) = X \backslash int(H)$ is ω-open. Thus, $int(H)$ is ω-closed.

(2)⇒(3): Let H be any subset of X. Then $X \backslash int_\omega(H)$ is ω-closed in X and by (2), $int(X \backslash int_\omega(H))$ is ω-closed in X. Therefore $cl(int_\omega(H))$ is ω-open in X and $cl(int_\omega(H)) \subset int_\omega(cl(H))$.

(3)⇒(4): Let H be semi-ω-open. Then $H \subset cl(int_\omega(H))$ and by (3), $H \subset int_\omega(cl(H))$. Thus, H is pre-$\omega$-open.

(4)⇒(5): Let H be a strong β-ω-open set. Then $H \subset cl(int_\omega(cl_\omega(H)))$ and $cl(H) \subset cl(cl(int_\omega(cl_\omega(H)))) = cl(int_\omega(cl_\omega(H))) \subset cl(int_\omega(cl(H)))$. Thus $cl(H)$ is semi-ω-open. By (4), $cl(H)$ is pre-ω-open. So $cl(H) \subset int_\omega(cl(cl(H))) = int_\omega(cl(H))$. Hence $cl(H)$ is ω-open.

(5)⇒(6): Let H be strong β-ω-open. By (5), $cl(H) = int_\omega(cl(H))$ and $H \subset cl(H)$. Hence $H \subset int_\omega(cl(H))$ and thus H is pre-ω-open.

(6)⇒(7): Let H be semi-ω-open. Since a semi-ω-open set is strong β-ω-open by Proposition 5.15 and by (6) it is pre-ω-open. Since H is semi-ω-open and pre-ω-open, by Theorem 2.11, H is α-ω-open.

Conversely, the result follows from the fact that every α-ω-open set is semi-ω-open.

(7)⇒(1): Let H be an ω-open set of X. Then $H = int_\omega(H)$ and $cl(H) = cl(int_\omega(H)) \subset cl(int_\omega(cl(H)))$. Thus $cl(H)$ is semi-ω-open and by (7), $cl(H)$ is α-ω-open. Therefore $cl(H) \subset int_\omega(cl(int_\omega(cl(H)))) = int_\omega(cl(H))$ and hence $cl(H) = int_\omega(cl(H))$. Hence $cl(H)$ is ω-open and X is ω-extremally disconnected. □

Theorem 5.19. *For an ω-extremally disconnected space (X, τ), the following are equivalent:*

(1). H is an ω-open.

(2). H is α-ω-open and a locally ω-closed.

(3). H is pre-ω-open and a locally ω-closed.

(4). H is semi-ω-open and a locally ω-closed.

(5). H is b-ω-open and a locally ω-closed.

Proof.
(1)⇒(2); (2)⇒(3); (2)⇒(4); (3)⇒(5) and (4)⇒(5): Obvious by Remark 2.10 and (1) of Remark 5.2.

(5)⇒(1): Since H is b-ω-open in X, it follows that $H \subset cl(int_\omega(H)) \cup int_\omega(cl(H))$. Since H is locally ω-closed, there exists an ω-open set G such that $H = G \cap cl(H)$ and $H \subset G$. It follows from Theorem 5.18(3) that $H \subset G \cap [cl(int_\omega(H)) \cup int_\omega(cl(H))] = [G \cap cl(int_\omega(H))] \cup [G \cap int_\omega(cl(H))] \subset [G \cap int_\omega(cl(H))] \cup [G \cap int_\omega(cl(H))] = G \cap int_\omega(cl(H)) = int_\omega(G) \cap int_\omega(cl(H)) = int_\omega(G \cap cl(H)) = int_\omega(H)$. Thus, $H = int_\omega(H)$ and hence H is ω-open in X. □

6. Decompositions of ω-continuity

Definition 6.1 ([3]). *A function $f : X \to Y$ is called pre-ω-continuous (resp. α-ω-continuous, ω-continuous) if $f^{-1}(V)$ is pre-ω-open (resp. α-ω-open, ω-open) in X for each open set V in Y.*

Definition 6.2 ([4]). *A function $f : X \to Y$ is called semi-ω-continuous if $f^{-1}(V)$ is semi-ω-open in X for each open set V in Y.*

Definition 6.3. *A function $f : X \to Y$ is called ω-$\mathcal{B}^{\star\star}$-continuous (resp. $\omega^{\star\star}$-$\mathcal{B}$-continuous, $\omega^{\star}$-$\mathcal{B}$-continuous, ω-$\mathcal{B}^{\star}$-continuous, $\mathcal{H}^{\star}_{\omega}$-continuous, ω-$\mathcal{AB}^{\#}$-continuous, contra locally ω-continuous) if $f^{-1}(V)$ is an ω-$\mathcal{B}^{\star\star}$-set (resp. an $\omega^{\star\star}$-$\mathcal{B}$-set, an $\omega^{\star}$-$\mathcal{B}$-set, an ω-$\mathcal{B}^{\star}$-set, a $\mathcal{H}^{\star}_{\omega}$-set, an ω-$\mathcal{AB}^{\#}$-set, a locally ω-closed set) in X for each open set V in Y.*

Theorem 6.4. *For a function $f : X \to Y$, the following are equivalent:*

(1). f is ω-continuous.

(2). f is semi-ω-continuous and ω-$\mathcal{B}^{\star\star}$-continuous.

Proof. This is an immediate consequence of Proposition 3.6. □

Theorem 6.5. *For a function $f : X \to Y$, the following are equivalent:*

(1). f is ω-continuous.

(2). f is semi-ω-continuous and $\omega^{\star\star}$-$\mathcal{B}$-continuous.

Proof. This is an immediate consequence of Proposition 3.15. □

Theorem 6.6. *For a function $f : X \to Y$, the following are equivalent:*

(1). f is ω-continuous.

(2). f is pre-ω-continuous and $\omega^{\star}$-$\mathcal{B}$-continuous.

Proof. This is an immediate consequence of Proposition 4.6. □

Theorem 6.7. *For a function $f : X \to Y$, the following are equivalent:*

(1). f is ω-continuous.

(2). f is pre-ω-continuous and ω-$\mathcal{B}^{\star}$-continuous.

Proof. This is an immediate consequence of Proposition 4.14. □

Theorem 6.8. *For a function $f : X \to Y$, the following are equivalent:*

(1). f is ω-continuous.

(2). f is α-ω-continuous and $\mathcal{H}^{\star}_{\omega}$-continuous.

(3). f is pre-ω-continuous and $\mathcal{H}^{\star}_{\omega}$-continuous.

Proof. This is an immediate consequence of Theorem 4.24. □

Theorem 6.9. *For a function $f : X \to Y$, the following are equivalent:*

(1). f is ω-continuous.

(2). f is α-ω-continuous and ω-$\mathcal{AB}^{\#}$-continuous.

(3). f is pre-ω-continuous and ω-$\mathcal{AB}^{\#}$-continuous.

Proof. This is an immediate consequence of Theorem 4.33. □

Theorem 6.10. *For a function $f : X \to Y$, the following are equivalent:*

(1). f is ω-continuous.

(2). f is pre-ω-continuous and contra locally ω-continuous.

Proof. This is an immediate consequence of Proposition 5.6. □

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