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$\mathcal{I}_g$-$\star$-closed Sets

Research Article

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Abstract: The notion of $\mathcal{I}_g$-$\star$-closed sets is introduced in ideal topological spaces. Characterizations and properties of $\mathcal{I}_g$-$\star$-closed sets and $\mathcal{I}_g$-$\star$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_g$-$\star$-open sets. Also, it is established that an $\mathcal{I}_g$-$\star$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

MSC: 54A05, 54A10.

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1. Introduction and Preliminaries

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If H⊆X, cl(H) and int(H) will, respectively, denote the closure and interior of H in (X, τ). A subset H of a space (X, τ) is called an α-open [15] (resp. semi-open [9], preopen [12]) set if H⊆int(cl(int(H))) (resp. H⊆cl(int(H)), H⊆int(cl(H))). The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The closure of H in (X, τ^α) is denoted by α-cl(H).

Definition 1.1 ([10]). *A subset H of a space (X, τ) is said to be*

(1). g-closed if cl(H)⊆U whenever H⊆U and U is open in X.

(2). g-open if its complement is g-closed.

An ideal $\mathcal{I}$ on a space (X, τ) is a nonempty collection of subsets of X which satisfies (i) A$\in\mathcal{I}$ and B⊆A⇒B$\in\mathcal{I}$ and (ii) A$\in\mathcal{I}$ and B$\in\mathcal{I}$⇒A∪B$\in\mathcal{I}$. Given a space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^*$: $\wp$(X)→$\wp$(X), called a local function [8] of A with respect to τ and $\mathcal{I}$, is defined as follows: for A⊆X, $A^*(\mathcal{I},\tau)$={x∈X | U∩A$\notin\mathcal{I}$ for every U∈τ(x)} where τ(x)={U∈τ | x∈U}. We will make use of the basic facts about the local functions [[6], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by $cl^*(A)=A\cup A^*(\mathcal{I},\tau)$ [17]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* for $\tau^*(\mathcal{I},\tau)$. $int^*(A)$ will denote the interior of A in (X, τ^*).
If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space or an ideal space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ).

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Definition 1.2. *A subset H of an ideal space $(X, \tau, \mathcal{I})$ is called $\star$-closed [6] (resp. $\star$-dense in itself [4]) if $H^*\subseteq H$ or $H = cl^*(H)$ (resp. $H\subseteq H^*$). The complement of a $\star$-closed set is called $\star$-open.*

Definition 1.3. *A subset H of an ideal topological space $(X, \tau, \mathcal{I})$ is called*

(1). $\mathcal{I}_g$-closed [1] if $H^*\subseteq U$ whenever $H\subseteq U$ and U is open in X.

(2). $\star$-g-closed [11] if $cl(H) \subseteq U$ whenever $H \subseteq U$ and U is $\star$-open in X.

Remark 1.4. *[11] For a subset of an ideal space $(X, \tau, \mathcal{I})$, we have the following implications:*

$$\textbf{closed} \longrightarrow \star\text{-}g\text{-}\textbf{closed} \longrightarrow g\text{-}\textbf{closed}$$

None of the above implications is reversible.

Lemma 1.5 ([6]). *Let $(X, \tau, \mathcal{I})$ be an ideal space and A, B subsets of X. Then the following properties hold:*

(1). $A\subseteq B\Rightarrow A^*\subseteq B^*$,

(2). $A^*=cl(A^*)\subseteq cl(A)$,

(3). $(A^*)^*\subseteq A^*$,

(4). $(A\cup B)^*=A^*\cup B^*$,

(5). $(A\cap B)^*\subseteq A^*\cap B^*$.

Definition 1.6. *An ideal $\mathcal{I}$ is said to be*

(1). codense [2] or τ-boundary [14] if $\tau \cap \mathcal{I}=\{\emptyset\}$,

(2). completely codense [2] if $PO(X) \cap \mathcal{I}=\{\emptyset\}$, where $PO(X)$ is the family of all preopen sets in (X, τ).

Lemma 1.7. *Every completely codense ideal is codense but not conversely [2].*

Lemma 1.8. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $H\subseteq X$. If $H\subseteq H^*$, then $H^*=cl(H^*)=cl(H) =cl^*(H)$ [[16], Theorem 5].*

Lemma 1.9. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is codense if and only if $G\subseteq G^*$ for every semi-open set G in X [[16], Theorem 3].*

Lemma 1.10. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $\mathcal{I}$ is completely codense, then $\tau^*\subseteq\tau^\alpha$ [[16], Theorem 6].*

Definition 1.11. *[1] An ideal space $(X,\tau,\mathcal{I})$ is called $T_\mathcal{I}$ if every $\mathcal{I}_g$-closed subset of X is $\star$-closed in X.*

Lemma 1.12. *If $(X, \tau, \mathcal{I})$ is a $T_\mathcal{I}$ ideal space and H is an $\mathcal{I}_g$-closed set, then H is a $\star$-closed set [[13], Corollary 2.2].*

Lemma 1.13. *Every g-closed set is $\mathcal{I}_g$-closed but not conversely [[1], Theorem 2.1].*

2. Properties of $\mathcal{I}_g$-$\star$-closed Sets

Definition 2.1. *A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be*

(1). $\mathcal{I}_g$-$\star$-closed if $A^*\subseteq U$ whenever $A\subseteq U$ and U is $\star$-open,

(2). $\mathcal{I}_g$-$\star$-open if its complement is $\mathcal{I}_g$-$\star$-closed.

Theorem 2.2. *If $(X, \tau, \mathcal{I})$ is any ideal space, then every $\mathcal{I}_g$-$\star$-closed set is $\mathcal{I}_g$-closed.*

Proof. It follows from the fact that every open set is $\star$-open. □

The converse of Theorem 2.2 is not true in general as shown in the following Example.

Example 2.3. *Let $X=\{a, b, c\}$, $\tau=\{\emptyset, X, \{c\}\}$ and $\mathcal{I}=\{\emptyset, \{a\}\}$. Then $\{b\}$ is $\mathcal{I}_g$-closed but not $\mathcal{I}_g$-$\star$-closed. $\star$-closed sets are $\emptyset$, X, $\{a\}$, $\{a, b\}$; $\star$-g-closed sets $=$ $\mathcal{I}_g$-$\star$-closed sets are $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$ and g-closed sets $=$ $\mathcal{I}_g$-closed sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$.*

Proposition 2.4. *If A is a $\star$-closed set of $(X, \tau, \mathcal{I})$ and B is closed in (X, τ), then $A \cap B$ is $\star$-closed in $(X, \tau, \mathcal{I})$.*

Proof. $cl^\star(A \cap B) \subseteq cl^\star(A) \cap cl^\star(B) \subseteq cl^\star(A) \cap cl(B) = A \cap B$. Hence $A \cap B = cl^\star(A \cap B)$ and $A \cap B$ is $\star$-closed. □

The following Theorem gives characterizations of $\mathcal{I}_g$-$\star$-closed sets.

Theorem 2.5. *If $(X, \tau, \mathcal{I})$ is any ideal space and $A \subseteq X$, then the following are equivalent.*

(1). A is $\mathcal{I}_g$-$\star$-closed,

(2). $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is $\star$-open in X,

(3). $cl^*(A) - A$ contains no nonempty $\star$-closed set,

(4). $A^* - A$ contains no nonempty $\star$-closed set.

Proof.
(1) ⇒ (2) Let A ⊆ U where U is $\star$-open in X. Since A is $\mathcal{I}_g$-$\star$-closed, $A^* \subseteq U$ and so $cl^*(A) = A \cup A^* \subseteq U$.
(2) ⇒ (3) Let F be a $\star$-closed subset such that $F \subseteq cl^*(A) - A$. Then $F \subseteq cl^*(A)$. Also $F \subseteq cl^*(A) - A \subseteq X - A$ and hence $A \subseteq X - F$ where $X - F$ is $\star$-open. By (2) $cl^*(A) \subseteq X - F$ and so $F \subseteq X - cl^*(A)$. Thus $F \subseteq cl^*(A) \cap X - cl^*(A) = \emptyset$.
(3) ⇒ (4) $A^* - A = A \cup A^* - A = cl^*(A) - A$ which has no nonempty $\star$-closed subset by (3).
(4) ⇒ (1) Let $A \subseteq U$ where U is $\star$-open. Then $X - U \subseteq X - A$ and so $A^* \cap (X - U) \subseteq A^* \cap (X - A) = A^* - A$. Since A^* is always a closed subset and $X - U$ is $\star$-closed, $A^* \cap (X - U)$ is a $\star$-closed set contained in $A^* - A$ and hence $A^* \cap (X - U) = \emptyset$ by (4). Thus $A^* \subseteq U$ and A is $\mathcal{I}_g$-$\star$-closed. □

Theorem 2.6. *Every $\star$-closed set is $\mathcal{I}_g$-$\star$-closed.*

Proof. Let A be a $\star$-closed set. To prove A is $\mathcal{I}_g$-$\star$-closed, let U be any $\star$-open set such that $A \subseteq U$. Since A is $\star$-closed, $A^* \subseteq A \subseteq U$. Thus A is $\mathcal{I}_g$-$\star$-closed. □

The converse of Theorem 2.6 is not true in general as shown in the following Example.

Example 2.7. *In Example 2.3, $\{a, c\}$ is $\mathcal{I}_g$-$\star$-closed but not $\star$-closed.*

Theorem 2.8. *Let $(X, \tau, \mathcal{I})$ be an ideal space. For every $A \in \mathcal{I}$, A is $\mathcal{I}_g$-$\star$-closed.*

Proof. Let $A \in \mathcal{I}$ and let $A \subseteq U$ where U is $\star$-open. Since $A \in \mathcal{I}$, $A^* = \emptyset \subseteq U$. Thus A is $\mathcal{I}_g$-$\star$-closed. □

Theorem 2.9. *If $(X, \tau, \mathcal{I})$ is an ideal space, then A^* is always $\mathcal{I}_g$-$\star$-closed for every subset A of X.*

Proof. Let $A^* \subseteq U$ where U is $\star$-open. Since $(A^*)^* \subseteq A^*$ [6], we have $(A^*)^* \subseteq U$. Hence A^* is $\mathcal{I}_g$-$\star$-closed. □

Theorem 2.10. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then every $\mathcal{I}_g$-$\star$-closed, $\star$-open set is $\star$-closed.*

Proof. Let A be $\mathcal{I}_g$-$\star$-closed and $\star$-open. We have $A \subseteq A$ where A is $\star$-open. Since A is $\mathcal{I}_g$-$\star$-closed, $A^* \subseteq A$. Thus A is $\star$-closed. □

Corollary 2.11. *If (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$ ideal space and A is an $\mathcal{I}_g$-$\star$-closed set, then A is a $\star$-closed set.*

Proof. By assumption A is $\mathcal{I}_g$-$\star$-closed in (X, τ, $\mathcal{I}$) and so by Theorem 2.2, A is $\mathcal{I}_g$-closed. Since (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$-space, by Definition 1.11, A is $\star$-closed. □

Corollary 2.12. *Let (X, τ, $\mathcal{I}$) be an ideal space and A be an $\mathcal{I}_g$-$\star$-closed set. Then the following are equivalent.*

(1). A is a $\star$-closed set,

(2). $cl^*(A) - A$ is a $\star$-closed set,

(3). $A^* - A$ is a $\star$-closed set.

Proof.
(1) $\Rightarrow$ (2) By (1) A is $\star$-closed. Hence $A^* \subseteq A$ and $cl^*(A) - A = (A \cup A^*) - A = \emptyset$ which is a $\star$-closed set.
(2) $\Rightarrow$ (3) $A^* - A = A \cup A^* - A = cl^*(A) - A$ which is a $\star$-closed set by (2).
(3) $\Rightarrow$ (1) Since A is $\mathcal{I}_g$-$\star$-closed, by Theorem 2.5 $A^* - A$ contains no non-empty $\star$-closed set. By assumption (3) $A^* - A$ is $\star$-closed and hence $A^* - A = \emptyset$. Thus $A^* \subseteq A$ and A is $\star$-closed. □

Theorem 2.13. *Let (X, τ, $\mathcal{I}$) be an ideal space. Then every $\star$-g-closed set is an $\mathcal{I}_g$-$\star$-closed set.*

Proof. Let A be a $\star$-g-closed set. Let U be any $\star$-open set such that $A \subseteq U$. Since A is $\star$-g-closed, $cl(A) \subseteq U$. So, $A^* \subseteq cl(A) \subseteq U$ and thus A is $\mathcal{I}_g$-$\star$-closed. □

The converse of Theorem 2.13 is not true in general as shown in the following Example.

Example 2.14. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then $\{a\}$ is $\mathcal{I}_g$-$\star$-closed but not $\star$-g-closed. g-closed sets are $\emptyset$, X, $\{b\}$, $\{a, b\}$, $\{b, c\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{b, c, d\}$; $\mathcal{I}_g$-$\star$-closed sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{d\}$, $\{a, b\}$, $\{a, d\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$ and $\star$-g-closed sets are $\emptyset$, X, $\{b\}$, $\{a, b\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$.*

Theorem 2.15. *If (X, τ, $\mathcal{I}$) is an ideal space and A is a $\star$-dense in itself, $\mathcal{I}_g$-$\star$-closed subset of X, then A is $\star$-g-closed.*

Proof. Let $A \subseteq U$ where U is $\star$-open. Since A is $\mathcal{I}_g$-$\star$-closed, $A^* \subseteq U$. As A is $\star$-dense in itself, by Lemma 1.8, $cl(A) = A^*$. Thus $cl(A) \subseteq U$ and hence A is $\star$-g-closed. □

Corollary 2.16. *If (X, τ, $\mathcal{I}$) is any ideal space where $\mathcal{I}=\{\emptyset\}$, then A is $\mathcal{I}_g$-$\star$-closed if and only if A is $\star$-g-closed.*

Proof. In (X, τ, $\mathcal{I}$), if $\mathcal{I} = \{\emptyset\}$ then $A^* = cl(A)$ for the subset A. A is $\mathcal{I}_g$-$\star$-closed $\Leftrightarrow$ $A^* \subseteq U$ whenever $A \subseteq U$ and U is $\star$-open $\Leftrightarrow$ $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\star$-open $\Leftrightarrow$ A is $\star$-g-closed. □

Corollary 2.17. *In an ideal space (X, τ, $\mathcal{I}$) where $\mathcal{I}$ is codense, if A is a semi-open and $\mathcal{I}_g$-$\star$-closed subset of X, then A is $\star$-g-closed.*

Proof. By Lemma 1.9, A is $\star$-dense in itself. By Theorem 2.15, A is $\star$-g-closed. □

Example 2.18. *In Example 2.3, $\{b\}$ is g-closed but not $\mathcal{I}_g$-$\star$-closed.*

Example 2.19. *In Example 2.14, $\{a\}$ is $\mathcal{I}_g$-$\star$-closed but not g-closed.*

Remark 2.20. *We see that from Examples 2.18 and 2.19, g-closed sets and $\mathcal{I}_g$-$\star$-closed sets are independent.*

Remark 2.21. *We have the following implications for the subsets stated above.*

$$\begin{tikzcd}
\text{closed} \arrow[r] \arrow[d] & \star\text{-}g\text{-closed} \arrow[r] \arrow[d] & g\text{-closed} \arrow[d] \\
\star\text{-closed} \arrow[r] & \mathcal{I}_g\text{-}\star\text{-closed} \arrow[r] & \mathcal{I}_g\text{-closed}
\end{tikzcd}$$

Theorem 2.22. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq X$. Then A is $\mathcal{I}_g$-$\star$-closed if and only if $A = F - N$ where F is $\star$-closed and N contains no nonempty $\star$-closed set.*

Proof. If A is $\mathcal{I}_g$-$\star$-closed, then by Theorem 2.5(4), $N = A^* - A$ contains no nonempty $\star$-closed set. If $F = cl^*(A)$, then F is $\star$-closed such that $F - N = (A \cup A^*) - (A^* - A) = (A \cup A^*) \cap (A^* \cap A^c)^c = (A \cup A^*) \cap ((A^*)^c \cup A) = (A \cup A^*) \cap (A \cup (A^*)^c) = A \cup (A^* \cap (A^*)^c) = A$.

Conversely, suppose $A = F - N$ where F is $\star$-closed and N contains no nonempty $\star$-closed set. Let U be an $\star$-open set such that $A \subseteq U$. Then $F - N \subseteq U$ which implies that $F \cap (X - U) \subseteq N$. Now $A \subseteq F$ and $F^* \subseteq F$ then $A^* \subseteq F^*$ and so $A^* \cap (X - U) \subseteq F^* \cap (X - U) \subseteq F \cap (X - U) \subseteq N$. Since $A^* \cap (X - U)$ is $\star$-closed, by hypothesis $A^* \cap (X - U) = \emptyset$ and so $A^* \subseteq U$. Hence A is $\mathcal{I}_g$-$\star$-closed. □

Theorem 2.23. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If A and B are subsets of X such that $A \subseteq B \subseteq cl^*(A)$ and A is $\mathcal{I}_g$-$\star$-closed, then B is $\mathcal{I}_g$-$\star$-closed.*

Proof. Since A is $\mathcal{I}_g$-$\star$-closed, then by Theorem 2.5(3), $cl^*(A) - A$ contains no nonempty $\star$-closed set. But $cl^*(B) - B \subseteq cl^*(A) - A$ and so $cl^*(B) - B$ contains no nonempty $\star$-closed set. Hence B is $\mathcal{I}_g$-$\star$-closed. □

Corollary 2.24. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If A and B are subsets of X such that $A \subseteq B \subseteq A^*$ and A is $\mathcal{I}_g$-$\star$-closed, then A and B are $\star$-g-closed sets.*

Proof. Let A and B be subsets of X such that $A \subseteq B \subseteq A^*$. Then $A \subseteq B \subseteq A^* \subseteq cl^*(A)$. Since A is $\mathcal{I}_g$-$\star$-closed, by Theorem 2.23, B is $\mathcal{I}_g$-$\star$-closed. Since $A \subseteq B \subseteq A^*$, we have $A^* = B^*$. Hence $A \subseteq A^*$ and $B \subseteq B^*$. Thus A is $\star$-dense in itself and B is $\star$-dense in itself and by Theorem 2.15, A and B are $\star$-g-closed. □

The following Theorem gives a characterization of $\mathcal{I}_g$-$\star$-open sets.

Theorem 2.25. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq X$. Then A is $\mathcal{I}_g$-$\star$-open if and only if $F \subseteq int^*(A)$ whenever F is $\star$-closed and $F \subseteq A$.*

Proof. Suppose A is $\mathcal{I}_g$-$\star$-open. If F is $\star$-closed and $F \subseteq A$, then $X - A \subseteq X - F$ and so $cl^*(X - A) \subseteq X - F$ by Theorem 2.5(2). Therefore $F \subseteq X - cl^*(X - A) = int^*(A)$. Hence $F \subseteq int^*(A)$.

Conversely, suppose the condition holds. Let U be an $\star$-open set such that $X - A \subseteq U$. Then $X - U \subseteq A$ and so $X - U \subseteq int^*(A)$. Therefore $cl^*(X - A) \subseteq U$. By Theorem 2.5(2), $X - A$ is $\mathcal{I}_g$-$\star$-closed. Hence A is $\mathcal{I}_g$-$\star$-open. □

Corollary 2.26. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq X$. If A is $\mathcal{I}_g$-$\star$-open, then $F \subseteq int^*(A)$ whenever F is closed and $F \subseteq A$.*

The following Theorem gives a property of $\mathcal{I}_g$-$\star$-closed.

Theorem 2.27. *Let $(X, \tau, \mathcal{I})$ be an ideal space and A, $B \subseteq X$. If A is $\mathcal{I}_g$-$\star$-open and $int^*(A) \subseteq B \subseteq A$, then B is $\mathcal{I}_g$-$\star$-open.*

Proof. Since $\mathrm{int}^*(A) \subseteq B \subseteq A$, we have $X - A \subseteq X - B \subseteq X - \mathrm{int}^*(A) = \mathrm{cl}^*(X - A)$. By assumption A is $\mathcal{I}_g$-$\star$-open and so $X - A$ is $\mathcal{I}_g$-$\star$-closed. Hence by Theorem 2.23, $X - B$ is $\mathcal{I}_g$-$\star$-closed and B is $\mathcal{I}_g$-$\star$-open. □

The following Theorem gives a characterization of $\mathcal{I}_g$-$\star$-closed sets in terms of $\mathcal{I}_g$-$\star$-open sets.

Theorem 2.28. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq X$. Then the following are equivalent.*

(1). A is $\mathcal{I}_g$-$\star$-closed,

(2). $A \cup (X - A^*)$ is $\mathcal{I}_g$-$\star$-closed,

(3). $A^* - A$ is $\mathcal{I}_g$-$\star$-open.

Proof.
(1) $\Rightarrow$ (2). Suppose A is $\mathcal{I}_g$-$\star$-closed. If U is any $\star$-open set such that $(A \cup (X - A^*)) \subseteq U$, then $X - U \subseteq X - (A \cup (X - A^*)) = [A \cup (A^*)^c]^c = A^* \cap A^c = A^* - A$. Since A is $\mathcal{I}_g$-$\star$-closed, by Theorem 2.5(4), it follows that $X - U = \emptyset$ and so $X = U$. Since X is the only $\star$-open set containing $A \cup (X - A^*)$, clearly, $A \cup (X - A^*)$ is $\mathcal{I}_g$-$\star$-closed.
(2) $\Rightarrow$ (1). Suppose $A \cup (X - A^*)$ is $\mathcal{I}_g$-$\star$-closed. If F is any $\star$-closed set such that $F \subseteq A^* - A = X - (A \cup (X - A^*))$, then $A \cup (X - A^*) \subseteq X - F$ and $X - F$ is $\star$-open. Therefore, $(A \cup (X - A^*))^* \subseteq X - F$ which implies that $A^* \cup (X - A^*)^* \subseteq X - F$ and so $F \subseteq X - A^*$. Since $F \subseteq A^*$, it follows that $F = \emptyset$. Hence A is $\mathcal{I}_g$-$\star$-closed.
The equivalence of (2) and (3) follows from the fact that $X - (A^* - A) = A \cup (X - A^*)$. □

Theorem 2.29. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then every subset of X is $\mathcal{I}_g$-$\star$-closed if and only if every $\star$-open set is $\star$-closed.*

Proof. Suppose every subset of X is $\mathcal{I}_g$-$\star$-closed. Let U be any $\star$-open in X. Then $U \subseteq U$ and U is $\mathcal{I}_g$-$\star$-closed by assumption implies $U^* \subseteq U$. Hence U is $\star$-closed.
Conversely, let $A \subseteq X$ and U be any $\star$-open such that $A \subseteq U$. Since U is $\star$-closed by assumption, we have $A^* \subseteq U^* \subseteq U$. Thus A is $\mathcal{I}_g$-$\star$-closed. □

The following Theorem gives a characterization of normal spaces in terms of $\mathcal{I}_g$-$\star$-open sets.

Theorem 2.30. *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

(1). X is normal,

(2). For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}_g$-$\star$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$,

(3). For any closed set A and open set V containing A, there exists an $\mathcal{I}_g$-$\star$-open set U such that $A \subseteq U \subseteq cl^*(U) \subseteq V$.

Proof.
(1)$\Rightarrow$(2) The proof follows from the fact that every open set is $\mathcal{I}_g$-$\star$-open.
(2)$\Rightarrow$(3) Suppose A is closed and V is an open set containing A. Since A and $X - V$ are disjoint closed sets, there exist disjoint $\mathcal{I}_g$-$\star$-open sets U and W such that $A \subseteq U$ and $X - V \subseteq W$. Since $X - V$ is $\star$-closed and W is $\mathcal{I}_g$-$\star$-open, $X - V \subseteq \mathrm{int}^*(W)$. Then $X - \mathrm{int}^*(W) \subseteq V$. Again $U \cap W = \emptyset$ which implies that $U \cap \mathrm{int}^*(W) = \emptyset$ and so $U \subseteq X - \mathrm{int}^*(W)$. Then $\mathrm{cl}^*(U) \subseteq X - \mathrm{int}^*(W) \subseteq V$ and thus U is the required $\mathcal{I}_g$-$\star$-open set with $A \subseteq U \subseteq \mathrm{cl}^*(U) \subseteq V$.
(3)$\Rightarrow$(1) Let A and B be two disjoint closed subsets of X. Then A is a closed set and $X - B$ an open set containing A. By hypothesis, there exists an $\mathcal{I}_g$-$\star$-open set U such that $A \subseteq U \subseteq \mathrm{cl}^*(U) \subseteq X - B$. Since U is $\mathcal{I}_g$-$\star$-open and A is $\star$-closed we have

A$\subseteq$int*(U). Since $\mathcal{I}$ is completely codense, by Lemma 1.10, $\tau^*\subseteq\tau^\alpha$ and so int*(U) and X$-$cl*(U)$\in\tau^\alpha$. Hence A$\subseteq$int*(U)$\subseteq$ int(cl(int(int*(U))))=G and B$\subseteq$X$-$cl*(U)$\subseteq$ int(cl(int(X$-$cl*(U))))=H. G and H are the required disjoint open sets containing A and B respectively, which proves (1). □

Definition 2.31. *A subset H of an ideal space $(X, \tau, \mathcal{I})$ is said to be an $g\alpha$-$\star$-closed if α-cl(H)$\subseteq$U whenever H$\subseteq$U and U is $\star$-open. The complement of an $g\alpha$-$\star$-closed set is called $g\alpha$-$\star$-open.*

If $\mathcal{I}=\mathcal{N}$, it is not difficult to see that $\mathcal{I}_g$-$\star$-closed sets coincide with $g\alpha$-$\star$-closed sets and so we have the following Corollary.

Corollary 2.32. *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}=\mathcal{N}$. Then the following are equivalent.*

(1). X is normal,

(2). For any disjoint closed sets A and B, there exist disjoint $g\alpha$-$\star$-open sets U and V such that A$\subseteq$U and B$\subseteq$V,

(3). For any closed set A and open set V containing A, there exists an $g\alpha$-$\star$-open set U such that A$\subseteq$U$\subseteq\alpha$-cl(U)$\subseteq$V.

Definition 2.33. *A subset H of an ideal space is said to be $\mathcal{I}$-compact [3] or compact modulo $\mathcal{I}$ [14] if for every open cover $\{U_\alpha \mid \alpha\in\Delta\}$ of H, there exists a finite subset Δ_0 of Δ such that $H-\cup\{U_\alpha \mid \alpha\in\Delta_0\}\in\mathcal{I}$. The space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-compact if X is $\mathcal{I}$-compact as a subset.*

Theorem 2.34. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If A is an $\mathcal{I}_g$-closed subset of X, then A is $\mathcal{I}$-compact [[13], Theorem 2.17].*

Corollary 2.35. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If A is an $\mathcal{I}_g$-$\star$-closed subset of X, then A is $\mathcal{I}$-compact.*

Proof. The proof follows from the fact that every $\mathcal{I}_g$-$\star$-closed set is $\mathcal{I}_g$-closed. □

3. $\star$-$\mathcal{I}$-locally Closed Sets

Definition 3.1. *A subset H of an ideal space $(X, \tau, \mathcal{I})$ is called a $\star$-$\mathcal{I}$-locally closed set (briefly, $\star$-$\mathcal{I}$-LC) if H=U$\cap$V where U is $\star$-open and V is $\star$-closed.*

Definition 3.2 ([7]). *A subset H of an ideal space $(X, \tau, \mathcal{I})$ is called a weakly $\mathcal{I}$-locally closed set (briefly, weakly $\mathcal{I}$-LC) if H=U$\cap$V where U is open and V is $\star$-closed.*

Proposition 3.3. *Let $(X, \tau, \mathcal{I})$ be an ideal space and H a subset of X. Then the following hold.*

(1). If H is $\star$-open, then H is $\star$-$\mathcal{I}$-LC-set.

(2). If H is $\star$-closed, then H is $\star$-$\mathcal{I}$-LC-set.

(3). If H is a weakly $\mathcal{I}$-LC-set, then H is a $\star$-$\mathcal{I}$-LC-set.

The converses of Proposition 3.3 are not true in general as shown in the following Examples.

Example 3.4.

(1). In Example 2.3, $\{b\}$ is a $\star$-$\mathcal{I}$-LC-set but it is not a $\star$-closed set.

(2). In Example 2.3, $\{a, b\}$ is a $\star$-$\mathcal{I}$-LC-set but it is not an $\star$-open set.

Example 3.5. *In Example 2.3, $\{b\}$ is a $\star$-$\mathcal{I}$-LC-set but it is not a weakly $\mathcal{I}$-LC-set.*

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If A is a $\star$-$\mathcal{I}$-LC-set and B is a $\star$-closed set, then A∩B is a $\star$-$\mathcal{I}$-LC-set.*

Proof. Let B be $\star$-closed, then A∩B=(U∩V)∩B=U∩(V∩B), where V∩B is $\star$-closed. Hence A∩B is a $\star$-$\mathcal{I}$-LC-set. □

Theorem 3.7. *A subset of an ideal space $(X, \tau, \mathcal{I})$ is $\star$-closed if and only if it is (i) weakly $\mathcal{I}$-LC and $\mathcal{I}_g$-closed [5] (ii) $\star$-$\mathcal{I}$-LC and $\mathcal{I}_g$-$\star$-closed.*

Proof. (ii) Necessity is trivial. We prove only sufficiency. Let A be $\star$-$\mathcal{I}$-LC-set and $\mathcal{I}_g$-$\star$-closed. Since A is $\star$-$\mathcal{I}$-LC, A=U∩V, where U is $\star$-open and V is $\star$-closed. So, we have A=U∩V⊆U. Since A is $\mathcal{I}_g$-$\star$-closed, $A^* \subseteq U$. Also since A = U∩V⊆V and V is $\star$-closed, we have $A^* \subseteq V$. Consequently, $A^* \subseteq$U∩V = A and hence A is $\star$-closed. □

Remark 3.8.

(1). The notions of weakly $\mathcal{I}$-LC-set and $\mathcal{I}_g$-closed set are independent [5].

(2). The notions of $\star$-$\mathcal{I}$-LC-set and $\mathcal{I}_g$-$\star$-closed set are independent.

Example 3.9. *In Example 2.3, $\{b\}$ is a $\star$-$\mathcal{I}$-LC-set but it is not an $\mathcal{I}_g$-$\star$-closed set.*

Example 3.10. *In Example 2.3, $\{a, c\}$ is an $\mathcal{I}_g$-$\star$-closed set but it is not a $\star$-$\mathcal{I}$-LC-set.*

4. Decompositions of $\star$-continuity

Definition 4.1. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be $\star$-continuous [5] (resp. $\mathcal{I}_g$-continuous [5], $\star$-$\mathcal{I}$-LC-continuous, $\mathcal{I}_g$-$\star$-continuous, weakly $\mathcal{I}$-LC-continuous [7]) if $f^{-1}(A)$ is $\star$-closed (resp. $\mathcal{I}_g$-closed, $\star$-$\mathcal{I}$-LC-set, $\mathcal{I}_g$-$\star$-closed, weakly $\mathcal{I}$-LC-set) in $(X, \tau, \mathcal{I})$ for every closed set A of (Y, σ).*

Theorem 4.2. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is $\star$-continuous if and only if it is (i) weakly $\mathcal{I}$-LC-continuous and $\mathcal{I}_g$-continuous [5]. (ii) $\star$-$\mathcal{I}$-LC-continuous and $\mathcal{I}_g$-$\star$-continuous.*

Proof. It is an immediate consequence of Theorem 3.7. □

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